Lecture 10: Duality

MATH 110-3

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July 6, 2023

Notation

Let V be a $\mathbb{F}$-vector space.

Linear Functionals

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- $\phi : \mathbb{F}^n \to \mathbb{F}$ where $\phi(x_1, \ldots, x_n) = c_1x_1 + \ldots + c_nx_n$
- $\phi : \mathcal{P}(\mathbb{R}) \to \mathbb{R}$ where $\phi(p) = \int_0^1 p dx$

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If $v_1, \ldots, v_n$ is a basis of V, then the dual basis of $v_1, \ldots, v_n$ is the list of $\phi_1, \ldots, \phi_n$ of V' such that

$$\phi_j(v_k) = \begin{cases} 1 & k = j, \\ 0 & k \neq j \end{cases}$$

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- $V = \mathbb{R}^2$, $B = \{(2,1), (3,1)$. Find the dual basis.

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Notice $(a_1\phi_1 + \ldots + a_n\phi_n)(v_j) = a_j$ for each $j \in [n]$. Thus, $a_j = 0$ for each j. □

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Then $D'(\phi)(p) = \phi(D(p)) = \phi(p') = \int_0^1 p' = p(1) - p(0)$.

Dual Map

Facts about Dual Map:

- $(S+T)' = S' + T'$ for all $S, T \in \mathcal{L}(V, W)$
- $(\lambda T)' = \lambda T'$ for all $\lambda \in \mathbb{F}$ and $T \in \mathcal{L}(V, W)$
- $(ST)' = T'S'$ for all $T \in \mathcal{L}(U, V)$ and all $S \in \mathcal{L}(V, W)$

Examples and Exercises

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Suppose ϕ_1, ϕ_2 denotes the dual basis of the standard basis of $\mathbb{R}^2$.

What are the linear functionals $T'(\phi_1)$ and $T'(\phi_2)$?

Lecture ended here!

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Describe $T'(\phi)$ on $\mathcal{P}(\mathbb{R})$.

Examples and Exercises

Suppose V is finite-dimensional and U is a subspace of V such that $U \neq V$. Prove that there exists $\phi \in V'$ such that $\phi(u) = 0$ for every $u \in U$ but $\phi \neq 0$.

Examples and Exercises

Suppose V is finite dimensional and $v \in V$ with $v \neq 0$. Prove that there exists $\phi \in V'$ such that $\phi(v) = 1$.

References

[Axl14] Sheldon Axler.
Linear Algebra Done Right.
Undergraduate Texts in Mathematics. Springer Cham, 2014.