Lecture 11: Duality (cont'd) and Rank

MATH 110-3

Franny Dean

July 10, 2023

Last time: Dual Space

Def'n:

We call $\mathcal{L}(V, \mathbb{F})$ the **dual space** of V and denote it V'.

Last time: Dual Space

Def'n:

We call $\mathcal{L}(V, \mathbb{F})$ the **dual space** of V and denote it V'.

Prop'n:

For V finite-dimensional, V' is also finite-dimensional and $\dim V' = \dim V$.

Dual Basis

Def'n:

If $v_1, \ldots, v_n$ is a basis of V, then the dual basis of $v_1, \ldots, v_n$ is the list of $\phi_1, \ldots, \phi_n$ of V' such that

$$\phi_j(v_k) = \begin{cases} 1 & k = j, \\ 0 & k \neq j \end{cases}$$

Dual Basis

Def'n:

If $v_1, \ldots, v_n$ is a basis of V, then the dual basis of $v_1, \ldots, v_n$ is the list of $\phi_1, \ldots, \phi_n$ of V' such that

$$\phi_j(v_k) = \begin{cases} 1 & k = j, \\ 0 & k \neq j \end{cases}$$

Examples:

- Dual basis of $e_1, \ldots, e_n$?

Dual Basis

Def'n:

If $v_1, \ldots, v_n$ is a basis of V, then the dual basis of $v_1, \ldots, v_n$ is the list of $\phi_1, \ldots, \phi_n$ of V' such that

$$\phi_j(v_k) = \begin{cases} 1 & k = j, \\ 0 & k \neq j \end{cases}$$

Examples:

- Dual basis of $e_1, \ldots, e_n$?
- Dual basis of $1, x, \ldots, x^m \in \mathcal{P}_m(\mathbb{R})$?

Dual Basis

Def'n:

If $v_1, \ldots, v_n$ is a basis of V, then the dual basis of $v_1, \ldots, v_n$ is the list of $\phi_1, \ldots, \phi_n$ of V' such that

$$\phi_j(v_k) = \begin{cases} 1 & k = j, \\ 0 & k \neq j \end{cases}$$

Examples:

- Dual basis of $e_1, \ldots, e_n$?
- Dual basis of $1, x, \ldots, x^m \in \mathcal{P}_m(\mathbb{R})$?

$$\frac{(x^k)^{(j)}(0)}{j!}$$

Last time: Dual Map

Def'n:

If $T \in \mathcal{L}(V, W)$, the **dual map** is the linear map $T' \in \mathcal{L}(W', V')$ defined $T'(\phi) = \phi \circ T$ for $\phi \in W'$.

Last time: Dual Map

Def'n:

If $T \in \mathcal{L}(V, W)$, the **dual map** is the linear map $T' \in \mathcal{L}(W', V')$ defined $T'(\phi) = \phi \circ T$ for $\phi \in W'$.

Today we will see the following results:

Prop'ns:

Suppose V, W are finite dimensional and $T \in \mathcal{L}(V, W)$. Then

- T is surjective if and only if T' is injective
- T is injective if and only if T' is surjective

Examples and Exercises

Define $T : \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$ by $(Tp)(x) = x^2 p(x)$ for $x \in \mathbb{R}$.

Examples and Exercises

Define $T : \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$ by $(Tp)(x) = x^2 p(x)$ for $x \in \mathbb{R}$.

Suppose $\phi \in \mathcal{P}(\mathbb{R})'$ defined by $\phi(p) = p''(7)$.

Examples and Exercises

Define $T : \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$ by $(Tp)(x) = x^2 p(x)$ for $x \in \mathbb{R}$.

Suppose $\phi \in \mathcal{P}(\mathbb{R})'$ defined by $\phi(p) = p''(7)$.

Describe $T'(\phi)$ on $\mathcal{P}(\mathbb{R})$.

Examples and Exercises

Suppose V is finite dimensional and $v \in V$ with $v \neq 0$. Prove that there exists $\phi \in V'$ such that $\phi(v) = 1$.

Annihilator

Def'n:

For $U \subset V$ the **annihilator** of U is denoted U^0 and is defined to be

$$U^0 = \{\phi \in V' : \phi(u) = 0 \text{ for all } u \in U\}.$$

Annihilator

Def'n:

For $U \subset V$ the **annihilator** of U is denoted U^0 and is defined to be

$$U^0 = \{\phi \in V' : \phi(u) = 0 \text{ for all } u \in U\}.$$

Fact [Axl14]:

U^0 is a subspace of V'.

Annihilator

Def'n:

For $U \subset V$ the **annihilator** of U is denoted U^0 and is defined to be

$$U^0 = \{\phi \in V' : \phi(u) = 0 \text{ for all } u \in U\}.$$

Fact [Axl14]:

U^0 is a subspace of V'.

Example:

Annihilator

Def'n:

For $U \subset V$ the **annihilator** of U is denoted U^0 and is defined to be

$$U^0 = \{\phi \in V' : \phi(u) = 0 \text{ for all } u \in U\}.$$

Fact [Axl14]:

U^0 is a subspace of V'.

Example:

$U \subset \mathcal{P}(\mathbb{R})$ where polynomials are multiples of x^2

Annihilator

Def'n:

For $U \subset V$ the **annihilator** of U is denoted U^0 and is defined to be

$$U^0 = \{\phi \in V' : \phi(u) = 0 \text{ for all } u \in U\}.$$

Fact [Axl14]:

U^0 is a subspace of V'.

Example:

$U \subset \mathcal{P}(\mathbb{R})$ where polynomials are multiples of x^2

$\phi \in \mathcal{P}(\mathbb{R})'$ defined as $\phi(p) = p'(0)$ is in U^0

Dimension of Annihilator

Prop'n:

V finite dimensional. U a subspace. Then

$$\dim U + \dim U^0 = \dim V.$$

Dimension of Annihilator

Prop'n:

V finite dimensional. U a subspace. Then

$$\dim U + \dim U^0 = \dim V.$$

Proof sketch.

Dimension of Annihilator

Prop'n:

V finite dimensional. U a subspace. Then

$$\dim U + \dim U^0 = \dim V.$$

Proof sketch.

Define $i \in \mathcal{L}(U, V)$ to be the inclusion map. Then i' is a linear map $V' \to U'$.

Dimension of Annihilator

Prop'n:

V finite dimensional. U a subspace. Then

$$\dim U + \dim U^0 = \dim V.$$

Proof sketch.

Define $i \in \mathcal{L}(U, V)$ to be the inclusion map. Then i' is a linear map $V' \to U'$.

$$\dim \operatorname{range} i' + \dim \operatorname{null} i' = \dim V' = \dim V$$

Dimension of Annihilator

Prop'n:

V finite dimensional. U a subspace. Then

$$\dim U + \dim U^0 = \dim V.$$

Proof sketch.

Define $i \in \mathcal{L}(U, V)$ to be the inclusion map. Then i' is a linear map $V' \to U'$.

$$\dim \operatorname{range} i' + \dim \operatorname{null} i' = \dim V' = \dim V$$

Using definitions, replace $\dim \operatorname{null} i'$ with $\dim U^0$.

Dimension of Annihilator

Prop'n:

V finite dimensional. U a subspace. Then

$$\dim U + \dim U^0 = \dim V.$$

Proof sketch.

Define $i \in \mathcal{L}(U, V)$ to be the inclusion map. Then i' is a linear map $V' \to U'$.

$$\dim \operatorname{range} i' + \dim \operatorname{null} i' = \dim V' = \dim V$$

Using definitions, replace $\dim \operatorname{null} i'$ with $\dim U^0$.
Need: $\dim \operatorname{range} i' = \dim U$...

Dimension of Annihilator

Prop'n:

V finite dimensional. U a subspace. Then

$$\dim U + \dim U^0 = \dim V.$$

Proof sketch.

Define $i \in \mathcal{L}(U, V)$ to be the inclusion map. Then i' is a linear map $V' \to U'$.

$$\dim \operatorname{range} i' + \dim \operatorname{null} i' = \dim V' = \dim V$$

Using definitions, replace $\dim \operatorname{null} i'$ with $\dim U^0$.
Need: $\dim \operatorname{range} i' = \dim U$...
$\operatorname{range} i' = U'$ because every linear functional of U can be extended to V

Null space of T'

Prop'n:

Suppose V and W are finite dimensional $T \in \mathcal{L}(V, W)$. Then

- $\text{null } T' = (\text{range } T)^0$
- $\dim \text{null } T' = \dim \text{null } T + \dim W - \dim V$

Null space of T'

Prop'n:

Suppose V and W are finite dimensional $T \in \mathcal{L}(V, W)$. Then

- null $T' = (\text{range } T)^0$
- $\dim \text{null } T' = \dim \text{null } T + \dim W - \dim V$

Proof notes.

Null space of T'

Prop'n:

Suppose V and W are finite dimensional $T \in \mathcal{L}(V, W)$. Then

- null $T' = (\text{range } T)^0$
- $\dim \text{null } T' = \dim \text{null } T + \dim W - \dim V$

Proof notes. part 1:

- null $T' \subseteq (\text{range } T)^0$
- null $T' \supseteq (\text{range } T)^0$

Null space of T'

Prop'n:

Suppose V and W are finite dimensional $T \in \mathcal{L}(V, W)$. Then

- null $T' = (\text{range } T)^0$
- $\dim \text{null } T' = \dim \text{null } T + \dim W - \dim V$

Proof notes. part 1:

- null $T' \subseteq (\text{range } T)^0$
- null $T' \supseteq (\text{range } T)^0$

part 2: Notice $\dim(\text{range } T)^0 = \dim W - \dim \text{range } T$.

T is surjective $\Leftrightarrow$ T' injective

Prop'n:

Suppose V and W are finite dimensional and $T \in \mathcal{L}(V, W)$. Then T is surjective if and only if T' is injective.

T is surjective $\Leftrightarrow$ T' injective

Prop'n:

Suppose V and W are finite dimensional and $T \in \mathcal{L}(V, W)$. Then T is surjective if and only if T' is injective.

Proof.

T is surjective ⇔ T' injective

Prop'n:

Suppose V and W are finite dimensional and $T \in \mathcal{L}(V, W)$. Then T is surjective if and only if T' is injective.

Proof. T is surjective $\Leftrightarrow$ range $T = W$

T is surjective $\Leftrightarrow$ T' injective

Prop'n:

Suppose V and W are finite dimensional and $T \in \mathcal{L}(V, W)$. Then T is surjective if and only if T' is injective.

Proof. T is surjective $\Leftrightarrow$ range $T = W$

range $T = W \Leftrightarrow (\text{range } T)^0 = 0$

T is surjective $\Leftrightarrow$ T' injective

Prop'n:

Suppose V and W are finite dimensional and $T \in \mathcal{L}(V, W)$. Then T is surjective if and only if T' is injective.

Proof. T is surjective $\Leftrightarrow$ range $T = W$

range $T = W \Leftrightarrow (\text{range } T)^0 = 0$

$(\text{range } T)^0 = 0 \Leftrightarrow \text{null } T' = 0$

Similar Results

Prop'n [Axl14]:

Suppose V and W are finite dimensional $T \in \mathcal{L}(V, W)$. Then

- $\dim \operatorname{range} T' = \dim \operatorname{range} T$
- $\operatorname{range} T' = (\operatorname{null} T)^0$

Similar Results

Prop'n [Axl14]:

Suppose V and W are finite dimensional $T \in \mathcal{L}(V, W)$. Then

- $\dim \operatorname{range} T' = \dim \operatorname{range} T$
- $\operatorname{range} T' = (\operatorname{null} T)^0$

Prop'n [Axl14]:

Suppose V and W are finite dimensional $T \in \mathcal{L}(V, W)$. Then T is injective if and only if T' is surjective.

Matrix Transpose

Def'n:

The **transpose** of a matrix A, denoted A^t is the matrix obtained by interchanging rows and columns.

Matrix Transpose

Def'n:

The **transpose** of a matrix A, denoted A^t is the matrix obtained by interchanging rows and columns.

$$(A^t)_{k,j} = A_{j,k}$$

Matrix Transpose

Def'n:

The **transpose** of a matrix A, denoted A^t is the matrix obtained by interchanging rows and columns.

$$(A^t)_{k,j} = A_{j,k}$$

Facts:

- $(A + C)^t = A^t + C^t$

Matrix Transpose

Def'n:

The **transpose** of a matrix A, denoted A^t is the matrix obtained by interchanging rows and columns.

$$(A^t)_{k,j} = A_{j,k}$$

Facts:

- $(A + C)^t = A^t + C^t$
- $(\lambda A)^t = \lambda A^t$

Matrix Transpose

Def'n:

The **transpose** of a matrix A, denoted A^t is the matrix obtained by interchanging rows and columns.

$$(A^t)_{k,j} = A_{j,k}$$

Facts:

- $(A + C)^t = A^t + C^t$
- $(\lambda A)^t = \lambda A^t$
- $(AC)^t = C^t A^t$

Matrix Transpose

Def'n:

The **transpose** of a matrix A, denoted A^t is the matrix obtained by interchanging rows and columns.

$$(A^t)_{k,j} = A_{j,k}$$

Facts:

- $(A + C)^t = A^t + C^t$
- $(\lambda A)^t = \lambda A^t$
- $(AC)^t = C^t A^t$
- For $T \in \mathcal{L}(V, W)$, $\mathcal{M}(T') = (\mathcal{M}(T))^t$.

Transpose is matrix of T'

Prop'n:

For $T \in \mathcal{L}(V, W)$, $\mathcal{M}(T') = (\mathcal{M}(T))^t$.

Transpose is matrix of T'

Prop'n:

For $T \in \mathcal{L}(V, W)$, $\mathcal{M}(T') = (\mathcal{M}(T))^t$.

Proof.

Transpose is matrix of T'

Prop'n:

For $T \in \mathcal{L}(V, W)$, $\mathcal{M}(T') = (\mathcal{M}(T))^t$.

Proof. Let $A = \mathcal{M}(T)$, $C = \mathcal{M}(T')$.

Transpose is matrix of T'

Prop'n:

For $T \in \mathcal{L}(V, W)$, $\mathcal{M}(T') = (\mathcal{M}(T))^t$.

Proof. Let $A = \mathcal{M}(T)$, $C = \mathcal{M}(T')$.

From definition, $T'(\psi_j) = \sum_{r=1}^{n} C_{r,j}\phi_r$ for some bases of W', V'.

Transpose is matrix of T'

Prop'n:

For $T \in \mathcal{L}(V, W)$, $\mathcal{M}(T') = (\mathcal{M}(T))^t$.

Proof. Let $A = \mathcal{M}(T)$, $C = \mathcal{M}(T')$.

From definition, $T'(\psi_j) = \sum_{r=1}^{n} C_{r,j}\phi_r$ for some bases of W', V'.

Rewrite by applying both sides to v_k

Transpose is matrix of T'

Prop'n:

For $T \in \mathcal{L}(V, W)$, $\mathcal{M}(T') = (\mathcal{M}(T))^t$.

Proof. Let $A = \mathcal{M}(T)$, $C = \mathcal{M}(T')$.

From definition, $T'(\psi_j) = \sum_{r=1}^{n} C_{r,j}\phi_r$ for some bases of W', V'.

Rewrite by applying both sides to v_k

$$(\psi_j \circ T)(v_k) = \sum_{r=1}^{n} C_{r,j}\phi_r(v_k) = C_{k,j}.$$

Transpose is matrix of T'

Prop'n:

For $T \in \mathcal{L}(V, W)$, $\mathcal{M}(T') = (\mathcal{M}(T))^t$.

Proof. Let $A = \mathcal{M}(T)$, $C = \mathcal{M}(T')$.

From definition, $T'(\psi_j) = \sum_{r=1}^{n} C_{r,j}\phi_r$ for some bases of W', V'.

Rewrite by applying both sides to v_k

$$(\psi_j \circ T)(v_k) = \sum_{r=1}^{n} C_{r,j}\phi_r(v_k) = C_{k,j}.$$

On the other hand, we also have

$$\begin{aligned}(\psi_j \circ T)(v_k) &= \psi_j(Tv_k) \\ &= \psi_j(\sum_{r=1}^{m} A_{r,k}w_r) \\ &= \sum_{r=1}^{m} A_{r,k}\psi_j(w_r)\end{aligned}$$

Matrix Rank

Def'n:

Suppose A is an $m \times n$ matrix with entries in $\mathbb{F}$.

- The ***row rank*** is the dimension of the span of the rows of A.
- The ***column rank*** is the dimension of the span of the columns of A.

Matrix Rank

Def'n:

Suppose A is an $m \times n$ matrix with entries in $\mathbb{F}$.

- The ***row rank*** is the dimension of the span of the rows of A.
- The ***column rank*** is the dimension of the span of the columns of A.

Examples:

Matrix Rank

Def'n:

Suppose A is an $m \times n$ matrix with entries in $\mathbb{F}$.

- The ***row rank*** is the dimension of the span of the rows of A.
- The ***column rank*** is the dimension of the span of the columns of A.

Examples:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 8 & 7 & 0 & 6 \end{pmatrix}$$

Range and Column Rank

Prop'n:

Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Then $\dim \operatorname{range} T$ is equal to the column rank of $\mathcal{M}(T)$.

Proof sketch. Pick $v_1, \ldots, v_n$ to be a basis of V.

$\mathcal{M} : \operatorname{span}(Tv_1, \ldots, Tv_n) \to \operatorname{span}(\mathcal{M}(Tv_1), \ldots, \mathcal{M}(Tv_n))$ is an isomorphism.

$\operatorname{range} T = \operatorname{span}(Tv_1, \ldots, Tv_n)$ and
$\dim \operatorname{span}(\mathcal{M}(Tv_1), \ldots, \mathcal{M}(Tv_n)) =$ column rank

Row Rank Equals Column Rank

Prop'n:

Suppose $A \in \mathbb{F}^{m,n}$. Then row rank of A equals the column rank of A.

Row Rank Equals Column Rank

Prop'n:

Suppose $A \in \mathbb{F}^{m,n}$. Then row rank of A equals the column rank of A.

Proof. Define $T : \mathbb{F}^{n,1} \to \mathbb{F}^{m,1}$ by $Tx = Ax$.

Row Rank Equals Column Rank

Prop'n:

Suppose $A \in \mathbb{F}^{m,n}$. Then row rank of A equals the column rank of A.

Proof. Define $T : \mathbb{F}^{n,1} \to \mathbb{F}^{m,1}$ by $Tx = Ax$. Then $\mathcal{M}(T) = A$.

Row Rank Equals Column Rank

Prop'n:

Suppose $A \in \mathbb{F}^{m,n}$. Then row rank of A equals the column rank of A.

Proof. Define $T : \mathbb{F}^{n,1} \to \mathbb{F}^{m,1}$ by $Tx = Ax$. Then $\mathcal{M}(T) = A$.

$$\begin{aligned}
\text{column rank of } A &= \text{column rank of } \mathcal{M}(T) \\
&= \dim \operatorname{range} T \\
&= \dim \operatorname{range} T' \\
&= \text{column rank of } \mathcal{M}(T') \\
&= \text{column rank of } A^t \\
&= \text{row rank of } A
\end{aligned}$$

Row Rank Equals Column Rank

Prop'n:

Suppose $A \in \mathbb{F}^{m,n}$. Then row rank of A equals the column rank of A.

Proof. Define $T : \mathbb{F}^{n,1} \to \mathbb{F}^{m,1}$ by $Tx = Ax$. Then $\mathcal{M}(T) = A$.

$$\begin{aligned} \text{column rank of } A &= \text{column rank of } \mathcal{M}(T) \\ &= \dim \text{range } T \\ &= \dim \text{range } T' \\ &= \text{column rank of } \mathcal{M}(T') \\ &= \text{column rank of } A^t \\ &= \text{row rank of } A \end{aligned}$$

□

Matrix Rank

Def'n:

We then say the **rank** of a matrix is the column rank (or equivalently the row rank).

References

[Axl14] Sheldon Axler.
Linear Algebra Done Right.
Undergraduate Texts in Mathematics. Springer Cham, 2014.