Lecture 12:
Invariant Subspaces, Eigenvectors

MATH 110-3

Franny Dean

July 11, 2023

Announcements

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- Only thing you need to know from this lecture is what an eigenvalue is, rank + things about dual maps from yesterdays class important but won't show up on exam−dual space will though

Motivation

Recall...

Def'n:

An **operator** is a linear map from a vector space to itself. The set of operators on V is denoted $\mathcal{L}(V)$.

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Suppose $V = U_1 \oplus \cdots \oplus U_m$.

We can describe $T \in \mathcal{L}(V)$ as $T|_{U_i}$.

⚠ But $T|_{U_i}$ might not be an operator on U_i ...

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First Examples: $\{0\}, V, \operatorname{null} T, \operatorname{range} T$

More Examples

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Let T be represented by $\begin{pmatrix} 2 & 3 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. I claim $U_1 = \text{span}(e_1, e_2)$ and $U_2 = \text{span}(e_3)$ are invariant subspaces.

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Def'n:

Suppose $T \in \mathcal{L}(V)$. A number $\lambda \in \mathbb{F}$ is called an **eigenvalue** of T if there exists $v \in V$ such that $v \neq 0$ and $Tv = \lambda v$. We then call v the corresponding **eigenvector**.

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Prop'n:

TFAE:

- λ is an eigenvalue
- $T - \lambda I$ is not injective
- $T - \lambda I$ is not surjective
- $T - \lambda I$ is not invertible

Example

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What are the eigenvalues and eigenvectors?

Over $\mathbb{R}$? None because 90 degree rotation.

Over $\mathbb{C}$? Solve for scalars λ that make $T(w, z) = \lambda(w, z)$.

Linear Independence of Eigenvectors

Prop'n:

Let $T \in \mathcal{L}(V)$. S'pose $\lambda_1, \ldots, \lambda_m$ are distinct eigenvalues of T and $v_1, \ldots, v_m$ are corresponding eigenvectors. Then $v_1, \ldots, v_m$ is linearly independent.

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- pick smallest k such that $v_k \in \operatorname{span}(v_1, \ldots, v_{k-1})$
- $v_k = a_1 v_1 + \ldots + a_{k-1} v_{k-1}$
- apply T: $\lambda_k v_k = a_1 \lambda_1 v_1 + \ldots + a_{k-1} \lambda_{k-1} v_{k-1}$
- subtract, get contradict on size of k

Number of Eigenvectors

Prop'n:

Suppose V is finite-dimensional. Then each operator on V has at most $\dim V$ distinct eigenvalues.

Polynomials Applied to Operators

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Def'n:

Suppose $T \in \mathcal{L}(V)$ and $p \in \mathcal{P}(\mathbb{F})$ is a polynomial

$$p(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_m z^m$$

for $z \in \mathbb{F}$. Then $p(T)$ is the *operator*

$$p(T) = a_0 + a_1 T + a_2 T^2 + \ldots + a_m T^m.$$

Product of Polynomials

$$(pq)(z) := p(z)q(z)$$

We have the following [Axl14]:

- $(pq)(T) = p(T)q(T)$
- $p(T)q(T) = q(T)p(T)$

Existence of Eigenvalues

Prop'n:

Every operator on a finite-dimensional, non-zero, complex vector space has an eigenvalue.

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Prop'n [Axl14]:

Each zero of a polynomial corresponds to a degree 1 factor, i.e. $p(\lambda) = 0$ if and only if $p(z) = (z - \lambda)q(z)$ for every $z \in \mathbb{F}$.

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Prop'n [Axl14]:

If $p \in \mathcal{P}(\mathbb{C})$ is a nonconstant polynomial, then p has a unique factorization up to reordering, of the form

$$p(z) = c(z - \lambda_1) \cdots (z - \lambda_m)$$

for $c, \lambda_i \in \mathbb{C}$.

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- $v, Tv, Tv^2, \ldots, T^n v$ are linearly dependent
- $0 = a_0 v + a_1 Tv + \ldots + a_n T^n v$ factors
- $0 = c(T - \lambda_1 I) \cdots (T - \lambda_m I)v$ implies one of the $T - \lambda_j I$ is not injective

References

[Axl14] Sheldon Axler.
Linear Algebra Done Right.
Undergraduate Texts in Mathematics. Springer Cham, 2014.