Lecture 13: Review

MATH 110-3

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Definitions

- Vector space
- Field
- Subspace
- $U + W$
- $U \oplus W$

Definitions

- Linear combination
- Span, spans
- $\mathcal{P}(\mathbb{R}), \mathcal{P}_n(\mathbb{R}), \mathbb{F}^S, \mathbb{F}^{[0,1]}$
- Linearly independent
- Bases
- Standard basis for $\mathbb{F}^n, \mathcal{P}(\mathbb{R})$
- Dimension

Definitions

- Linear maps
- $\mathcal{L}(V, W)$
- Null space
- Range
- Injective
- Surjective
- Matrix of a linear map $\mathcal{M}(T)$
- $\mathbb{F}^{n,m}$
- Matrix multiplication
- Invertible
- Isomorphism
- Operator

Definitions

- Linear functional
- V'
- U^0
- Dual map
- Dual basis
- Rank
- Eigenvalue, eigenvector
- Invariant subspace

Key results and tools to review

- Subspace criteria
- Conditions for direct sum
- Linear dependence lemma
- Lengths of linearly independent vs. spanning lists
- Turning linearly independent lists and spanning lists into bases
- Basis of domain
- Relationship between null space and injective and range and surjective
- Fundamental Theorem of Linear Maps (Rank-Nullity)
- Invertible = injective + surjective
- Dimension and isomorphic $\mathbb{F}$-vector spaces

Midterm Structure

- 4 questions
- First question: T/F, circle answers
- Second is computational - think bases and matrices
- Third, fourth are more abstract proofs

Review of Subspace Criteria

Give an example of a subset for each of the following vector spaces. Can we prove it is a subspace?

- $\mathbb{R}^{[0,1]}$
- $\mathcal{P}_3(\mathbb{R})$
- $\mathbb{C}^2$

Is the intersection of two subspaces a subspace? Is the union?

Is the dual space a subspace?

Review of Proving Lists for Bases

- Prove that the list $1, (x-2), (x-2)^2, (x-2)^3, (x-2)^4, (x-2)^5$ forms a basis of $\mathcal{P}_5(\mathbb{R})$.
- Find a basis for the subspace of $\mathcal{P}_5(\mathbb{R})$ defined as

$$W := \{p \in \mathcal{P}_5(\mathbb{R}) | p'(2) = 0\}$$

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Review of Matrices

- Find a matrix for the linear map $D : \mathcal{P}_5(\mathbb{R}) \to \mathcal{P}_4(\mathbb{R})$ defined $Dp = p'$ with respect to the following bases:
  - Standard basis of both spaces
  - $1, (x-2), (x-2)^2, (x-2)^3, (x-2)^4, (x-2)^5$, Standard basis of output space
  - What would the matrix look like if we considered the map to be $D : \mathcal{P}_5(\mathbb{R}) \to \mathcal{P}_5(\mathbb{R})$?

Linear Functionals

- Prove that every linear functional is either surjective or the zero map.
- Define $T : \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$ by $(Tp)(x) = x^2 p(x) + p''(x)$ for $x \in \mathbb{R}$.
  (a) Suppose $\phi \in \mathcal{P}(\mathbb{R})'$ is defined by $\phi(p) = p'(4)$. Describe the linear functional $T'(\phi)$ on $\mathcal{P}(\mathbb{R})$.
  (b) Suppose $\phi \in \mathcal{P}(\mathbb{R})'$ is defined by $\phi(p) = \int_0^1 p(x)dx$. Evaluate $(T'(\phi))(x^3)$.

Homework Problems to Review

- (Last problem Homework 4) Suppose V is finite-dimensional and $v_1, \ldots, v_m \in V$. Define a linear map $\Gamma : V' \to \mathbb{F}^m$ by

$$\Gamma(\phi) = (\phi(v_1), \ldots, \phi(v_m)).$$

  Prove that $v_1, \ldots, v_m$ spans V if and only if Γ is injective.

- (Last problem Homework 3) Suppose V and W are finite-dimensional with $\dim V \geq \dim W \geq 2$. Show that

$$\{T \in \mathcal{L}(V, W) : T \text{ is not surjective } \}$$

  is not a subspace of $\mathcal{L}(V, W)$.

References

[Axl14] Sheldon Axler.
Linear Algebra Done Right.
Undergraduate Texts in Mathematics. Springer Cham, 2014.