Lecture 14: Upper Triangular and Diagonal Matrices

MATH 110-3

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July 17, 2023

Announcements

- Midterm expository write up: 2+ pages, explain all of the theory to someone who has not taken the class, one question, any number of points lost

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- Midterm expository write up: 2+ pages, explain all of the theory to someone who has not taken the class, one question, any number of points lost
- Roughly overall grades, including homework + quizzes:
  - A: 90% and above
  - B: 80-89%
  - C: 67-79%
- Remember homework + quizzes + discussion worth 50% and exams worth 50%

Recall Eigenvectors

One Dimensional Invariant Subspaces

$$U = \{\lambda v : \lambda \in \mathbb{F}\}$$

$$Tv = \lambda v$$

Def'n:

Suppose $T \in \mathcal{L}(V)$. A number $\lambda \in \mathbb{F}$ is called an **eigenvalue** of T if there exists $v \in V$ such that $v \neq 0$ and $Tv = \lambda v$. We then call v the corresponding **eigenvector**.

Facts we learned:

Prop'n:

Let $T \in \mathcal{L}(V)$. S'pose $\lambda_1, \ldots, \lambda_m$ are distinct eigenvalues of T and $v_1, \ldots, v_m$ are corresponding eigenvectors. Then $v_1, \ldots, v_m$ is linearly independent.

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Suppose V is finite-dimensional. Then each operator on V has at most $\dim V$ distinct eigenvalues.

Prop'n:

Every operator on a finite-dimensional, non-zero, complex vector space has an eigenvalue.

Matrix of Operators

Def'n:

$T \in \mathcal{L}(V)$ and $v_1, \ldots, v_n$ is a basis for V.
We define $\mathcal{M}(T)$ to be the matrix whose entries are defined by

$$Tv_k = A_{1,k}v_1 + \ldots + A_{n,k}v_n.$$

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The **diagonal** of a square matrix is the entries along the line from the upper left corner to the bottom right corner.

Def'n:

A matrix is called **upper triangular** if all the entries below the diagonal equal 0.

Conditions for Upper-Triangular Matrix

Prop'n:

Suppose $T \in \mathcal{L}(V)$ and $v_1, \ldots, v_n$ is a basis of V. TFAE:

(a) the matrix of T with respect to $v_1, \ldots, v_n$ is upper triangular

(b) $Tv_j \in \text{span}(v_1, \ldots, v_j)$ for each $j = 1, \ldots, n$

(c) $\text{span}(v_1, \ldots, v_j)$ is invariant under T for each $j = 1, \ldots, n$

Proof.

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Proof.

- (a) $\Leftrightarrow$ (b) by definition
- (b) $\implies$ (c): $Tv_1 \in \text{span}(v_1) \subset \text{span}(v_1, \ldots, v_j)$

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$$Tv_j \in \text{span}(v_1, \ldots, v_j)$$

- (c) $\implies$ (b): $\text{span}(v_1, \ldots, v_j)$ invariant implies $Tv_j \in \text{span}(v_1, \ldots, v_j)$

Every Operator has an Upper-Triangular Form over $\mathbb{C}$

Prop'n:

Suppose V is a finite-dimensional $\mathbb{C}$-vector space and $T \in \mathcal{L}(V)$. Then T has an upper triangular matrix with respect to some basis of V.

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- $T - \lambda I$ is not surjective. So $U = \text{range}\,(T - \lambda I)$ is a proper subset of V.

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- U is invariant under T: $Tu = (T - \lambda I)u + \lambda u$
- $T|_U$ is an operator with an upper triangular matrix.
- Let $u_1, \ldots, u_n$ be a basis of U. Extend to basis of V:

$$u_1, \ldots, u_n, v_1, \ldots, v_m$$

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Proof every operator has upper triangular (Cont'd).

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- $Tv_k = (T - \lambda I)v_k + \lambda v_k$ implies

$$Tv_k \in \operatorname{span}(u_1, \ldots, u_m, v_1, \ldots, v_k)$$

Utility of Upper-Triangular Form

Prop'n [Axl14]:

Suppose $T \in \mathcal{L}(V)$ has an upper triangular matrix with respect to some basis of V. Then T is invertible if and only if all the entries on the diagonal of the upper-triangular matrix are nonzero.

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Example:

$$\begin{pmatrix} 2 & 3 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 7 \end{pmatrix}$$

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Example:

$$\begin{pmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

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Example:

$$T(x, y) = (41x + 7y, -20x + 74y)$$

With respect to the basis $(1, 4), (7, 5)$ the matrix is

$$\begin{pmatrix} 69 & 0 \\ 0 & 46 \end{pmatrix}$$

Conditions for Diagonalizability

Prop'n:

V finite-dimensional, $T \in \mathcal{L}(V)$ and $\lambda_1, \ldots, \lambda_m$ ***distinct*** eigenvalues. TFAE:

(a) T is diagonalizable

(b) V has a basis of eigenvectors of T

(c) there exist 1-dimensional subspaces $U_1, \ldots, U_n$ of V each invariant under T such that $V = U_1 \oplus \cdots \oplus U_n$

(d) $V = E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_m, T)$

(e) $\dim V = \dim E(\lambda_1, T) + \ldots + \dim E(\lambda_m, T)$

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WTS: (b) $\implies$ (d) $\implies$ (e) $\implies$ (b)

Proof (Cont'd).

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S'pose (b) again:

Then $V = E(\lambda_1, T) + \ldots + E(\lambda_m, T)$. Show the only way to write $0 = u_1 + \ldots + u_m$ is $u_i = 0$.

Proof (Cont'd).

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S'pose (d): True fact we didn't prove.

S'pose (e):Pick vectors $v_1, \ldots, v_n$ combining bases for each eigenspace. Show these are linearly independent …

Corollary

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If $T \in \mathcal{L}(V)$ has $\dim V$ distinct eigenvalues, then T is diagonalizable.

References

[Axl14] Sheldon Axler.
Linear Algebra Done Right.
Undergraduate Texts in Mathematics. Springer Cham, 2014.