Lecture 15: Inner Products and Norms

MATH 110-3

Franny Dean

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Motivation

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- Norm is not linear so...

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The **inner product** generalizes the dot product.

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- define the norm as $||z|| = \sqrt{|z_1|^2 + \ldots + |z_n|^2}$
- define inner product of $w = (w_1, \ldots, w_n)$ and $z = (z_1, \ldots, z_n)$ as

$$w_1\bar{z_1} + \ldots + w_n\bar{z_n}$$

Inner Product

Def'n:

An **inner product** on V is a function that takes each ordered pair (u, v) of elements of V to a number $\langle u, v\rangle \in \mathbb{F}$ and has the following properties:

positivity

$\langle v, v\rangle \geq 0$ for all $v \in V$;

definiteness

$\langle v, v\rangle = 0$ if and only if $v = 0$;

additivity in first slot

$\langle u + v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$ for all $u, v, w \in V$;

homogeneity in first slot

$\langle \lambda u, v\rangle = \lambda\langle u, v\rangle$ for all $\lambda \in \mathbf{F}$ and all $u, v \in V$;

conjugate symmetry

$\langle u, v\rangle = \overline{\langle v, u\rangle}$ for all $u, v \in V$.

Examples

- The **Euclidean inner product on** $\mathbb{F}^n$ is defined

$$\langle w_1, \ldots, w_n), (z_1, \ldots, z_n)\rangle = w_1\bar{z_1} + \ldots + w_n\bar{z_n}.$$

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- An inner product on the vector space of real-valued functions on $[-1, 1]$, $\mathbb{R}^{[-1,1]}$, defined

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)dx.$$

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- An inner product on $\mathcal{P}(\mathbb{R})$ could be

$$\langle p, q\rangle = \int_{0}^{\infty} p(x)q(x)e^{-x}dx.$$

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- $\mathbb{F}^n$ with the Euclidean inner product (assume this one)

Basic Properties

6.7 Basic properties of an inner product

(a) For each fixed $u \in V$, the function that takes v to $\langle v, u \rangle$ is a linear map from V to $\mathbf{F}$.

(b) $\langle 0, u \rangle = 0$ for every $u \in V$.

(c) $\langle u, 0 \rangle = 0$ for every $u \in V$.

(d) $\langle u, v + w \rangle = \langle u, v \rangle + \langle u, w \rangle$ for all $u, v, w \in V$.

(e) $\langle u, \lambda v \rangle = \bar{\lambda} \langle u, v \rangle$ for all $\lambda \in \mathbf{F}$ and $u, v \in V$.

Norms

A general inner product allows us to define for any inner product...

Def'n:

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- $\mathbb{R}^{[-1,1]}$ with inner product defined as previously,

$$||f|| = \sqrt{\int_{-1}^{1} (f(x))^2 dx}.$$

Basic Properties

6.10 Basic properties of the norm

Suppose $v \in V$.

(a) $\|v\| = 0$ if and only if $v = 0$.

(b) $\|\lambda v\| = |\lambda| \, \|v\|$ for all $\lambda \in \mathbf{F}$.

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Prop'n 6.12 [Axl14]:

- 0 is orthogonal to every vector in V
- 0 is the only vector orthogonal to itself

Pythagorean Theorem

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Proof.

$$\begin{aligned} ||u+v||^2 &= \langle u+v, u+v\rangle \\ &= \langle u,u\rangle + \langle u,v\rangle + \langle v,u\rangle + \langle v,v\rangle \\ &= ||u||^2 + ||v||^2 \end{aligned}$$

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So let

$$c = \frac{\langle u, v \rangle}{||v||^2}.$$

Orthogonal Decomposition

Prop'n:

Suppose $u, v \in V$ with $v \neq 0$. Set $c = \frac{\langle u, v \rangle}{||v||^2}$ and $w = u - \frac{\langle u, v \rangle}{||v||^2} v$. Then

$$\langle w, v \rangle = 0$$

and

$$u = cv + w.$$

Cauchy-Schwarz

Prop'n:

Suppose $u, v \in V$. Then

$$|\langle u, v\rangle| \leq ||u|| ||v||.$$

Equality is reached if and only if one of u or v is a scalar multiple of the other.

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- Orthogonal decomposition
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$$\begin{aligned}
||u+v||^2 &= \langle u+v, u+v \rangle \\
&= \langle u, u \rangle + \langle v, v \rangle + \langle u, v \rangle + \langle v, u \rangle \\
&= \langle u, u \rangle + \langle v, v \rangle + \langle u, v \rangle + \overline{\langle u, v \rangle} \\
&= ||u||^2 + ||v||^2 + 2\mathrm{Re}\langle u, v \rangle \\
&\leq ||u||^2 + ||v||^2 + 2|\langle u, v \rangle| \\
&\leq ||u||^2 + ||v||^2 + 2||u||||v|| \\
&= (||u|| + ||v||)^2
\end{aligned}$$

Triangle Inequality (Cont'd)

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Proof (Cont'd).

Notice equality holds if and only if $\langle u, v \rangle = ||u||||v||$.

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Using Cauchy-Inequality, true **iff** u, v are *non-negative* scalar multiples of each other.

Latex Asides

$\langle u, v \rangle$ is coded langle u, v rangle

$\bar{\lambda}$ is bar{lambda}

$\overline{\langle u, v \rangle}$ is overline{langle u,v rangle}

References

[Axl14] Sheldon Axler.
Linear Algebra Done Right.
Undergraduate Texts in Mathematics. Springer Cham, 2014.