Lecture 16: Orthonormal Bases

MATH 110-3

Franny Dean

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Inner Products and Norms

- Example
- Cauchy-Schwarz Inequality

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Def'n:

Two vectors $u, v \in V$ are orthogonal if $\langle u, v \rangle = 0$.

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- $(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}), (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0), (\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}})$

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Def'n:

An **orthonormal basis** of V is an orthonormal list of vectors in V that is also a basis of V.

Orthonormal Lists are Nice Because...

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Prop'n:

If $e_1, \ldots, e_m$ is an orthonormal list of vectors in V, then

$$||a_1 e_1 + \ldots + a_m e_m||^2 = |a_1|^2 + \ldots + |a_m|^2$$

for all $a_1, \ldots, a_m \in \mathbb{F}$.

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Every orthonormal list of vectors is linearly independent.

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Every orthonormal list of vectors is linearly independent.

Proof. What does the previous result say about the following a_i?

$$a_1 e_1 + \ldots + a_m e_m = 0$$

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Prop'n:

S'pose $e_1, \ldots, e_m$ is an orthonormal basis of V and $v \in V$. Then

$$v = \langle v, e_1 \rangle e_1 + \ldots + \langle v, e_n \rangle e_n$$

and

$$||v||^2 = |\langle v, e_1 \rangle|^2 + \ldots + |\langle v, e_n \rangle|^2.$$

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Proof.

$$v = a_1 e_1 + \ldots + a_n e_n$$

Take the inner product of both sides with each e_i.

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Algorithm [Axl14]

Inputs: $v_1, \ldots, v_m$ a list of linearly independent vectors in V

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How:

- $e_1 = \dfrac{v_1}{||v_1||}$

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How:

- $e_1 = \dfrac{v_1}{||v_1||}$
- $e_j = \dfrac{v_j - \langle v_j, e_1 \rangle e_1 - \ldots - \langle v_j, e_{j-1} \rangle e_{j-1}}{||v_j - \langle v_j, e_1 \rangle e_1 - \ldots - \langle v_j, e_{j-1} \rangle e_{j-1}||}$

Example

On $\mathcal{P}_2(\mathbb{R})$ consider the inner product given by

$$\langle p, q \rangle = \int_{-1}^{1} p(x)q(x)dx.$$

Apply Gram-Schmidt to $1, x, x^2$ to get an orthonormal basis.

Consequences of Gram-Schmidt

Prop'n 6.34:

Every finite-dimensional inner product space has an orthonormal basis.

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Prop'n 6.35:

Suppose V is finite-dimensional. Then every orthonormal list of vectors in V can be extended to an orthonormal basis of V.

Schur's Theorem

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S'pose V is a finite dimensional $\mathbb{C}$-vector space. Then $T \in \mathcal{L}(V)$ has an upper triangular matrix with respect to some orthonormal basis.

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Criteria for Upper-Triangular Matrix:

S'pose $T \in \mathcal{L}(V)$ and $v_1, \ldots, v_n$ is a basis of V. TFAE:

- the matrix of T with respect to $v_1, \ldots, v_n$ is upper triangular
- $\operatorname{span}(v_1, \ldots, v_j)$ is invariant under T for each $j = 1, \ldots, n$

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- Gram-Schmidt fixes the span

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If $u \in V$, then $v \to \langle v, u \rangle$ is a linear functional on V.

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Riesz Representation Theorem

S'pose V is finite dimensional and ϕ is a linear functional on V. Then there is a unique $u \in V$ such that

$$\phi(v) = \langle v, u \rangle$$

for every $v \in V$.

Proof of Riesz Representation Theorem

Existence:

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$$
\begin{aligned}
\phi(v) &= \phi(\langle v, e_1 \rangle e_1 + \ldots + \langle v, e_n \rangle e_n) \\
&= \langle v, e_1 \rangle \phi(e_1) + \ldots + \langle v, e_n \rangle \phi(e_n) \\
&= \langle v, \overline{\phi(e_1)} e_1 + \ldots + \overline{\phi(e_n)} e_n \rangle
\end{aligned}
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Suppose there are two u_1, u_2.

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Suppose there are two u_1, u_2.

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Suppose there are two u_1, u_2.

Then $\phi(v) = \langle v, u_1\rangle = \langle v, u_2\rangle$.

$0 = \langle v, u_1\rangle - \langle v, u_2\rangle = \langle v, u_1 - u_2\rangle$ for every v.

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Uniqueness:

Suppose there are two u_1, u_2.

Then $\phi(v) = \langle v, u_1\rangle = \langle v, u_2\rangle$.

$0 = \langle v, u_1\rangle - \langle v, u_2\rangle = \langle v, u_1 - u_2\rangle$ for every v. Implies $u_1 - u_2 = 0$.

Break

Discussion Questions

1. Suppose $T \in \mathcal{L}(V)$ and $T^2 = I$ and -1 is not an eigenvalue of T. Prove that $T = I$.
2. Suppose $T \in \mathcal{L}(\mathbb{F}^5)$ and $\dim E(8, T) = 4$. Prove that $T - 2I$ or $T - 6I$ is invertible.
3. Find $T \in \mathcal{L}(\mathbb{C}^3)$ such that 6 and 7 are eigenvalues of T and such that T does not have a diagonal matrix with respect to any basis of $\mathbb{C}^3$.
4. Show that the function that takes the pair of $\mathbb{R}^3$ vectors $((x_1, x_2, x_3), (y_1, y_2, y_3))$ to $x_1 y_1 + x_3 y_3$ is not an inner product on $\mathbb{R}^3$.

Discussion Questions

5. Suppose u, v are nonzero vectors in $\mathbb{R}^2$. Prove that $\langle u, v \rangle = ||u|| ||v|| \cos \theta$, where θ is the angle between u and v. *Hint:* Draw the triangle formed by $u, v, u - v$ and envoke the law of cosines.
6. Suppose $\theta \in \mathbb{R}$. Show that $(\cos \theta, \sin \theta), (-\sin \theta, \cos \theta)$ and $(\cos \theta, \sin \theta), (\sin \theta, -\cos \theta)$ are orthonormal bases of $\mathbb{R}^2$.
7. Suppose $T \in \mathcal{L}(\mathbb{R}^3)$ has an upper triangular matrix with respect to the basis $(1, 0, 0), (1, 1, 1), (1, 1, 2)$. Find an orthonormal basis of $\mathbb{R}^3$ with respect to which T has an upper triangular matrix.
8. Prove that if V is a real innner product space, then for all u, v,

$$\langle u, v \rangle = \frac{||u + v||^2 - ||u - v||^2}{4}.$$

Discussion Question Hints/Solutions

1. Write $T^2 = I$ as $(T - I)(T + I)v = 0$ for all v. Then if -1 is not an eigenvalue then $T + I$ is injective, thus, $(T - I)v = 0$ for all v and $T = I$.
2. The sum of the dimensions of the eigenspaces is less than or equal to $\dim \mathbb{F}^5 = 5$. Thus, at least one of $E(2, T)$ and $E(6, T)$ must be zero-dimensional implying that either 2 or 6 respectively is not an eigenvalue.
3. $T(x, y, z) = (6x + y, 6y, 7z)$ works. We needed a matrix with $\dim E(6, T) = \dim E(7, T) = 1$ and no other eigenvalues.
4. One property that fails is that $\langle (0, 1, 0), (0, 1, 0) \rangle = 0$.

Discussion Question Hints/Solutions

5. Law of cosines: $||u-v||^2 = ||u||^2 + |||v||^2 - 2||u||||v|| \cos\theta$. Using norm/inner product: $||u-v||^2 = \langle u-v, u-v\rangle = \ldots = ||u||^2 + ||v||^2 - 2\langle u, v\rangle$. Equate to get desired formula.
6. Must show that each vector has norm 1 and that pairs are orthogonal.
7. Apply Gram-Schmidt to the given basis. Solution: $(1,0,0), (0, \sqrt{2}/2, \sqrt{2}/2), (0, -\sqrt{2}/2, \sqrt{2}/2)$.
8. $$\begin{aligned} \frac{||u+v||^2 - ||u-v||^2}{4} &= \frac{\langle u+v, u+v\rangle - \langle u-v, u-v\rangle}{4} \\ &= \frac{||u||^2 + 2\langle u, v\rangle + ||v||^2 - (||u||^2 - 2\langle u, v\rangle + ||v||^2)}{4} \\ &= \frac{4\langle u, v\rangle}{4} = \langle u, v\rangle \end{aligned}$$

References

[Axl14] Sheldon Axler.
Linear Algebra Done Right.
Undergraduate Texts in Mathematics. Springer Cham, 2014.