Lecture 17: Orthogonal Complements and Minimization Problems

MATH 110-3

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July 20, 2023

Recall!

Important things from last lecture:

Def'n:

A list is **orthonormal** is each vector in the list has norm 1 and is orthogonal to every other vector. (Often denoted $e_1, \ldots, e_m$).

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Vector as linear combo of orthonormal basis has a formula:

$e_1, \ldots, e_n$ is an orthonormal basis, $v \in V$:

$$v = \langle v, e_1 \rangle e_1 + \ldots + \langle v, e_n \rangle e_n$$

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Let $U = \{(x, 3x, 0) | x \in \mathbb{R}\}$. Find $U^\perp$. We can calculate
$U^\perp = \{(y, -1/3y, z) | z, y \in \mathbb{R}\}$.

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Proof of 1. Use properties of dot product, subspace criteria.

Proof of 5. S'pose $v \in W^\perp$. Then $\langle v, u \rangle = 0$ for all $u \in W$ and so also all $u \in U \subset W$.

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$w = v - \langle v, e_1 \rangle e_1 - \ldots - \langle v, e_m \rangle e_m$. Show w is orthogonal to each e_j.
Then $w \in U^{\perp}$.

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Then $w \in U^{\perp}$.

Step 2: Prove direct sum.

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First $U \subseteq (U^{\perp})^{\perp}$:

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- $v \in (U^\perp)^\perp$. Write $v = u + w \in U \oplus U^\perp$ because $U \subset (U^\perp)^\perp$.

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Def'n:

S'pose U is a finite-dimensional subspace of V. The **orthogonal projection** of V *onto* U is the operator $P_U \in \mathcal{L}(V)$ defined as follows:

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How do we know this is well defined? $V = U \oplus U^{\perp}$.

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Generalizing... If $x \in V$ and $x \neq 0$ and $U = \text{span}(x)$. Then

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Generalizing... If $x \in V$ and $x \neq 0$ and $U = \text{span}(x)$. Then

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Why?

$$v = \frac{\langle v, x\rangle}{||x||^2} x + (v - \frac{\langle v, x\rangle}{||x||^2} x)$$

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Suppose u a finite-dimensional subspace of V and $v \in V$:

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Suppose u a finite-dimensional subspace of V and $v \in V$:

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Minimizing distance to a subspace:

S'pose U a finite-dimensional subspace of V, $v \in V$, $u \in U$. Then

$$||v - P_U v|| \leq ||v - u||.$$

Further the inequality is equality if and only if $u = P_U v$.

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Should get $P_U(4,5,6) = (\frac{9}{2}, \frac{9}{2}, 6)$.

Example 2

Find a polynomial $u(x)$ with real coefficients and degree at most 5 that approximates $\sin x$ as well as possible on the interval $[-\pi, \pi]$ in the sense that

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Let $U = \mathcal{P}_5(\mathbb{R})$ and find $u \in U$ such that $||\sin(x) - u||$ is as small as possible.

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- $u(x) = .987862x - .155271x^3 + .00564312x^5$

References

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