Lecture 18: Adjoints, Self-Adjoint, Normal

MATH 110-3

Franny Dean

July 24, 2023

Notation

Today V, W are finite-dimensional $\mathbb{C}$ or $\mathbb{R}$-vector spaces.

Back to Linear Maps and Operators!

Back to Linear Maps and Operators!

Def'n:

S'pose $T \in \mathcal{L}(V, W)$. The **adjoint** of T is the function $T^* : W \to V$ such that

$$\langle Tv, w \rangle = \langle v, T^* w \rangle$$

for every $v \in V$ and $w \in W$.

Back to Linear Maps and Operators!

Def'n:

S'pose $T \in \mathcal{L}(V, W)$. The **adjoint** of T is the function $T^* : W \to V$ such that

$$\langle Tv, w \rangle = \langle v, T^*w \rangle$$

for every $v \in V$ and $w \in W$.

*This is the *Hermitian* adjoint.

Back to Linear Maps and Operators!

Def'n:

S'pose $T \in \mathcal{L}(V, W)$. The **adjoint** of T is the function $T^* : W \to V$ such that

$$\langle Tv, w \rangle = \langle v, T^* w \rangle$$

for every $v \in V$ and $w \in W$.

*This is the *Hermitian* adjoint.

Why/how...?

Back to Linear Maps and Operators!

Def'n:

S'pose $T \in \mathcal{L}(V, W)$. The **adjoint** of T is the function $T^* : W \to V$ such that

$$\langle Tv, w \rangle = \langle v, T^* w \rangle$$

for every $v \in V$ and $w \in W$.

*This is the *Hermitian* adjoint.

Why/how...?

- Fix $w \in W$. Consider the linear functional $\mathcal{L}(V, \mathbb{F})$ that sends v to $\langle Tv, w \rangle$.

Back to Linear Maps and Operators!

Def'n:

S'pose $T \in \mathcal{L}(V, W)$. The **adjoint** of T is the function $T^* : W \to V$ such that

$$\langle Tv, w \rangle = \langle v, T^* w \rangle$$

for every $v \in V$ and $w \in W$.

*This is the *Hermitian* adjoint.

Why/how...?

- Fix $w \in W$. Consider the linear functional $\mathcal{L}(V, \mathbb{F})$ that sends v to $\langle Tv, w \rangle$.
- By the Riesz Representation Thm, there is a unique element of V such that this linear functional is given by $\langle v, \text{ that vector } \rangle$.

Back to Linear Maps and Operators!

Def'n:

S'pose $T \in \mathcal{L}(V, W)$. The **adjoint** of T is the function $T^* : W \to V$ such that

$$\langle Tv, w \rangle = \langle v, T^* w \rangle$$

for every $v \in V$ and $w \in W$.

*This is the *Hermitian* adjoint.

Why/how...?

- Fix $w \in W$. Consider the linear functional $\mathcal{L}(V, \mathbb{F})$ that sends v to $\langle Tv, w \rangle$.
- By the Riesz Representation Thm, there is a unique element of V such that this linear functional is given by $\langle v, \text{ that vector } \rangle$.
- $T^* w$ is *that vector*

Example

Given $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined $T(x_1, x_2, x_3) = (x_2 + 3x_3, 2x_1)$. Find a formula for T^*.

Example

Given $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined $T(x_1, x_2, x_3) = (x_2 + 3x_3, 2x_1)$. Find a formula for T^*.

Start with

$$\langle (x_1, x_2, x_3), T^*(y_1, y_2) \rangle = \langle T(x_1, x_2, x_3), (y_1, y_2) \rangle$$

Example

Given $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined $T(x_1, x_2, x_3) = (x_2 + 3x_3, 2x_1)$. Find a formula for T^*.

Start with

$$\langle (x_1, x_2, x_3), T^*(y_1, y_2) \rangle = \langle T(x_1, x_2, x_3), (y_1, y_2) \rangle$$

Solution: $T^*(y_1, y_2) = (2y_2, y_1, 3y_1)$.

Adjoint is a Linear Map

Prop'n:

If $T \in \mathcal{L}(V, W)$, then $T^* \in \mathcal{L}(W, V)$.

Adjoint is a Linear Map

Prop'n:

If $T \in \mathcal{L}(V, W)$, then $T^* \in \mathcal{L}(W, V)$.

Proof.

Adjoint is a Linear Map

Prop'n:

If $T \in \mathcal{L}(V, W)$, then $T^* \in \mathcal{L}(W, V)$.

Proof. Additivity:

Adjoint is a Linear Map

Prop'n:

If $T \in \mathcal{L}(V, W)$, then $T^* \in \mathcal{L}(W, V)$.

Proof. Additivity:

$$\begin{aligned}
\langle v, T^*(w_1 + w_2) \rangle &= \langle Tv, w_1 + w2 \rangle \\
&= \langle Tv, w_1 \rangle + \langle Tv, w2 \rangle \\
&= \langle v, T^* w_1 \rangle + \langle v, T^* w_2 \rangle \\
&= \langle v, T^* w_1 + T^* w_2 \rangle
\end{aligned}$$

Adjoint is a Linear Map

Prop'n:

If $T \in \mathcal{L}(V, W)$, then $T^* \in \mathcal{L}(W, V)$.

Proof. Additivity:

$$\begin{aligned}\langle v, T^*(w_1 + w_2)\rangle &= \langle Tv, w_1 + w2\rangle \\ &= \langle Tv, w_1\rangle + \langle Tv, w2\rangle \\ &= \langle v, T^*w_1\rangle + \langle v, T^*w_2\rangle \\ &= \langle v, T^*w_1 + T^*w_2\rangle\end{aligned}$$

Homogeneity also uses the proof technique of *flipping* T^* from one side of the inner product to be T on the other side.

Adjoint Properties

Prop'n [Axl14]:

1. $(S+T)^* = S^* + T^*$

Adjoint Properties

Prop'n [Axl14]:

1. $(S+T)^* = S^* + T^*$
2. $(\lambda T)^* = \bar{\lambda} T^*$

Adjoint Properties

Prop'n [Axl14]:

1. $(S+T)^* = S^* + T^*$
2. $(\lambda T)^* = \bar{\lambda} T^*$
3. $(T^*)^* = T$

Adjoint Properties

Prop'n [Axl14]:

1. $(S+T)^* = S^* + T^*$
2. $(\lambda T)^* = \bar{\lambda} T^*$
3. $(T^*)^* = T$
4. $I^* = I$

Adjoint Properties

Prop'n [Axl14]:

1. $(S+T)^* = S^* + T^*$
2. $(\lambda T)^* = \bar{\lambda} T^*$
3. $(T^*)^* = T$
4. $I^* = I$
5. $(ST)^* = T^* S^*$

Adjoint Properties

Prop'n [Axl14]:

1. $(S + T)^* = S^* + T^*$
2. $(\lambda T)^* = \bar{\lambda} T^*$
3. $(T^*)^* = T$
4. $I^* = I$
5. $(ST)^* = T^* S^*$

Let's prove 1,3.

Null space and range of T^*

Prop'n [Axl14]:

$T \in \mathcal{L}(V, W)$

1. null $T^* = (\text{range } T)^\perp$
2. range $T^* = (\text{null } T)^\perp$
3. null $T = (\text{range } T^*)^\perp$
4. range $T = (\text{null } T^*)^\perp$

Null space and range of T^*

Prop'n [Axl14]:

$T \in \mathcal{L}(V, W)$

1. null $T^* = (\text{range } T)^\perp$
2. range $T^* = (\text{null } T)^\perp$
3. null $T = (\text{range } T^*)^\perp$
4. range $T = (\text{null } T^*)^\perp$

Proof.

Null space and range of T^*

Prop'n [Axl14]:

$T \in \mathcal{L}(V, W)$

1. null $T^* = (\text{range } T)^\perp$
2. range $T^* = (\text{null } T)^\perp$
3. null $T = (\text{range } T^*)^\perp$
4. range $T = (\text{null } T^*)^\perp$

Proof.

$$\begin{aligned} w \in \text{null } T^* &\Leftrightarrow T^*w = 0 \\ &\Leftrightarrow \langle v, T^*w \rangle = 0 \text{ for all } v \in V \\ &\Leftrightarrow \langle Tv, w \rangle = 0 \text{ for all } v \in V \\ &\Leftrightarrow w \in (\text{range } T)^\perp \end{aligned}$$

Null space and range of T^*

Prop'n [Axl14]:

$T \in \mathcal{L}(V, W)$

1. null $T^* = (\text{range } T)^\perp$
2. range $T^* = (\text{null } T)^\perp$
3. null $T = (\text{range } T^*)^\perp$
4. range $T = (\text{null } T^*)^\perp$

Proof.

$$\begin{aligned} w \in \text{null } T^* &\Leftrightarrow T^* w = 0 \\ &\Leftrightarrow \langle v, T^* w \rangle = 0 \text{ for all } v \in V \\ &\Leftrightarrow \langle Tv, w \rangle = 0 \text{ for all } v \in V \\ &\Leftrightarrow w \in (\text{range } T)^\perp \end{aligned}$$

How can we conclude the rest of the proof?

Matrix of T^*

Prop'n:

$T \in \mathcal{L}(V, W)$. Orthonormal bases, $e_1, \ldots, e_n$ of V and $f_1, \ldots, f_m$ of W. Then $\mathcal{M}(T^*)$ with respect to these bases is the **conjugate transpose** of $\mathcal{M}(T)$ with these bases.

Matrix of T^*

Prop'n:

$T \in \mathcal{L}(V, W)$. Orthonormal bases, $e_1, \ldots, e_n$ of V and $f_1, \ldots, f_m$ of W. Then $\mathcal{M}(T^*)$ with respect to these bases is the **conjugate transpose** of $\mathcal{M}(T)$ with these bases.

What's the conjugate transpose?

Matrix of T^*

Prop'n:

$T \in \mathcal{L}(V, W)$. Orthonormal bases, $e_1, \ldots, e_n$ of V and $f_1, \ldots, f_m$ of W. Then $\mathcal{M}(T^*)$ with respect to these bases is the **conjugate transpose** of $\mathcal{M}(T)$ with these bases.

What's the conjugate transpose?

- Transpose

Matrix of T^*

Prop'n:

$T \in \mathcal{L}(V, W)$. Orthonormal bases, $e_1, \ldots, e_n$ of V and $f_1, \ldots, f_m$ of W. Then $\mathcal{M}(T^*)$ with respect to these bases is the **conjugate transpose** of $\mathcal{M}(T)$ with these bases.

What's the conjugate transpose?

- Transpose
- Take the complex conjugate of each entry

Matrix of T^*

Prop'n:

$T \in \mathcal{L}(V, W)$. Orthonormal bases, $e_1, \ldots, e_n$ of V and $f_1, \ldots, f_m$ of W. Then $\mathcal{M}(T^*)$ with respect to these bases is the **conjugate transpose** of $\mathcal{M}(T)$ with these bases.

What's the conjugate transpose?

- Transpose
- Take the complex conjugate of each entry

Example:

Matrix of T^*

Prop'n:

$T \in \mathcal{L}(V, W)$. Orthonormal bases, $e_1, \dots, e_n$ of V and $f_1, \dots, f_m$ of W. Then $\mathcal{M}(T^*)$ with respect to these bases is the **conjugate transpose** of $\mathcal{M}(T)$ with these bases.

What's the conjugate transpose?

- Transpose
- Take the complex conjugate of each entry

Example: The complex conjugate of $\begin{pmatrix} 2 & 3+4i & 7-i \\ 3 & 1 & 1-11i \end{pmatrix}$ is

$$\begin{pmatrix} 2 & 3 \\ 3-4i & 1 \\ 7+i & 1+11i \end{pmatrix}$$

Proof of Matrix of T^*

Prop'n:

$T \in \mathcal{L}(V, W)$. Orthonormal bases, $e_1, \ldots, e_n$ of V and $f_1, \ldots, f_m$ of W. Then $\mathcal{M}(T^*)$ with respect to these bases is the **conjugate transpose** of $\mathcal{M}(T)$ with these bases.

Proof of Matrix of T^*

Prop'n:

$T \in \mathcal{L}(V, W)$. Orthonormal bases, $e_1, \ldots, e_n$ of V and $f_1, \ldots, f_m$ of W. Then $\mathcal{M}(T^*)$ with respect to these bases is the **conjugate transpose** of $\mathcal{M}(T)$ with these bases.

Proof.

Proof of Matrix of T^*

Prop'n:

$T \in \mathcal{L}(V, W)$. Orthonormal bases, $e_1, \ldots, e_n$ of V and $f_1, \ldots, f_m$ of W. Then $\mathcal{M}(T^*)$ with respect to these bases is the **conjugate transpose** of $\mathcal{M}(T)$ with these bases.

Proof.

Because the bases are orthonormal, we can read the entries of the columns of $\mathcal{M}(T)$ off of

$$Te_k = \langle Te_k, f_1\rangle f_1 + \ldots \langle Te_k, f_m\rangle f_m$$

and $\mathcal{M}(T)_{j,k} = \langle Te_k, f_j\rangle$.

Proof of Matrix of T^*

Prop'n:

$T \in \mathcal{L}(V, W)$. Orthonormal bases, $e_1, \ldots, e_n$ of V and $f_1, \ldots, f_m$ of W. Then $\mathcal{M}(T^*)$ with respect to these bases is the **conjugate transpose** of $\mathcal{M}(T)$ with these bases.

Proof.

Because the bases are orthonormal, we can read the entries of the columns of $\mathcal{M}(T)$ off of

$$Te_k = \langle Te_k, f_1\rangle f_1 + \ldots \langle Te_k, f_m\rangle f_m$$

and $\mathcal{M}(T)_{j,k} = \langle Te_k, f_j\rangle$. We have that the columns of $\mathcal{M}(T^*)$ come from

$$T^*f_k = \langle T^*f_k, e_1\rangle e_1 + \ldots \langle T^*f_k, e_n\rangle e_n$$

Proof of Matrix of T^*

Prop'n:

$T \in \mathcal{L}(V, W)$. Orthonormal bases, $e_1, \ldots, e_n$ of V and $f_1, \ldots, f_m$ of W. Then $\mathcal{M}(T^*)$ with respect to these bases is the **conjugate transpose** of $\mathcal{M}(T)$ with these bases.

Proof.

Because the bases are orthonormal, we can read the entries of the columns of $\mathcal{M}(T)$ off of

$$Te_k = \langle Te_k, f_1\rangle f_1 + \ldots \langle Te_k, f_m\rangle f_m$$

and $\mathcal{M}(T)_{j,k} = \langle Te_k, f_j\rangle$. We have that the columns of $\mathcal{M}(T^*)$ come from

$$T^*f_k = \langle T^*f_k, e_1\rangle e_1 + \ldots \langle T^*f_k, e_n\rangle e_n$$

and $\mathcal{M}(T^*)_{j,k} = \langle T^*f_k, e_j\rangle = \langle f_k, Te_j\rangle = \overline{\langle Te_j, f_k\rangle} = \overline{\mathcal{M}(T)_{k,j}}$

What are the differences between T^* and T'?

- $T^* : W \to V$, $T' : W' \to V'$

What are the differences between T^* and T'?

- $T^* : W \to V$, $T' : W' \to V'$
- Matrix of T^* is the conjugate transpose, Matrix of T' is the transpose

Self-Adjoint

Now consider operators,

Self-Adjoint

Now consider operators,

Def'n:

An operator $T \in \mathcal{L}(V)$ is called **self-adjoint** if $T = T^*$.

Self-Adjoint

Now consider operators,

Def'n:

An operator $T \in \mathcal{L}(V)$ is called **self-adjoint** if $T = T^*$.
This is definition of *Hermitian*.

Why care about Self-Adjoint/Hermitian operators?

Prop'n:

Every eigenvalue of a self-adjoint operator is real.

Why care about Self-Adjoint/Hermitian operators?

Prop'n:

Every eigenvalue of a self-adjoint operator is real.

Proof.

Why care about Self-Adjoint/Hermitian operators?

Prop'n:

Every eigenvalue of a self-adjoint operator is real.

Proof. S'pose $Tv = \lambda v$.

Why care about Self-Adjoint/Hermitian operators?

Prop'n:

Every eigenvalue of a self-adjoint operator is real.

Proof. S'pose $Tv = \lambda v$.

$$\lambda ||v||^2 = \langle \lambda v, v \rangle = \langle Tv, v \rangle = \langle v, Tv \rangle = \langle v, \lambda v \rangle = \bar{\lambda} ||v||^2$$

Why care about Self-Adjoint/Hermitian operators?

Prop'n:

Every eigenvalue of a self-adjoint operator is real.

Proof. S'pose $Tv = \lambda v$.

$$\lambda||v||^2 = \langle \lambda v, v \rangle = \langle Tv, v \rangle = \langle v, Tv \rangle = \langle v, \lambda v \rangle = \bar{\lambda}||v||^2$$

$$\lambda = \bar{\lambda}$$

Other Results

Prop'n [Axl14]:

Over $\mathbb{C}$, $\langle Tv, v\rangle$ is real for all v if and only if $T = T^*$.

Other Results

Prop'n [Axl14]:

Over $\mathbb{C}$, $\langle Tv, v\rangle$ is real for all v if and only if $T = T^*$.

Prop'n [Axl14]:

Over $\mathbb{C}$, Tv is orthogonal to v for all v if and only if $T = 0$.

Prop'n [Axl14]:

If $T = T^*$ and Tv is orthogonal to v for all v, then $T = 0$.

Normal Operators

Def'n:

An operator T on an inner product space is called **normal** if $TT^* = T^*T$.

Normal Operators

Def'n:

An operator T on an inner product space is called **normal** if $TT^* = T^*T$.

$\begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix}$ is not self-adjoint but is normal.

Why care about normal operators?

Prop'n 7.20:

An operator T is normal if and only if $||Tv|| = ||T^*v||$ for all v.

Prop'n 7.21:

A normal operator T and its adjoint share the same eigenvectors.

Prop'n 7.22:

The eigenvectors of a normal operator of distinct eigenvalues are orthogonal.

Proofs

Prop'n 7.20:

An operator T is normal if and only if $||Tv|| = ||T^*v||$ for all v.

Proofs

Prop'n 7.20:

An operator T is normal if and only if $||Tv|| = ||T^*v||$ for all v.

Proof.

Proofs

Prop'n 7.20:

An operator T is normal if and only if $||Tv|| = ||T^*v||$ for all v.

Proof.

$$\begin{aligned} T \text{ is normal } &\Leftrightarrow T^*T - TT^* = 0 \\ &\Leftrightarrow \langle (T^*T - TT^*)v, v\rangle = 0 \text{ for all } v \in V \\ &\Leftrightarrow \langle T^*Tv, v\rangle = \langle TT^*v, v\rangle \\ &\Leftrightarrow \langle Tv, Tv\rangle = \langle T^*v, T^*v\rangle \\ &\Leftrightarrow ||Tv||^2 = ||T^*v||^2 \end{aligned}$$

In the second line, we need that $T^*T - TT^*$ is self-adjoint.

Proofs

Prop'n 7.21:

A normal operator T and its adjoint share the same eigenvectors.

Proofs

Prop'n 7.21:

A normal operator T and its adjoint share the same eigenvectors.

Proof.

Proofs

Prop'n 7.21:

A normal operator T and its adjoint share the same eigenvectors.

Proof.

First, T normal implies $T - \lambda I$ is normal.

Proofs

Prop'n 7.21:

A normal operator T and its adjoint share the same eigenvectors.

Proof.

First, T normal implies $T - \lambda I$ is normal.

Then

$$0 = ||(T - \lambda I)v|| = ||(T - \lambda I)^* v|| = ||(T^* - \bar{\lambda} I)v||.$$

Proofs

Prop'n 7.22:

The eigenvectors of a normal operator of distinct eigenvalues are orthogonal.

Proofs

Prop'n 7.22:

The eigenvectors of a normal operator of distinct eigenvalues are orthogonal.

Proof.

Proofs

Prop'n 7.22:

The eigenvectors of a normal operator of distinct eigenvalues are orthogonal.

Proof.

S'pose α, β are distinct eigenvalues, $Tu = \alpha u$ and $Tv = \beta v$.

Proofs

Prop'n 7.22:

The eigenvectors of a normal operator of distinct eigenvalues are orthogonal.

Proof.

S'pose α, β are distinct eigenvalues, $Tu = \alpha u$ and $Tv = \beta v$.

$$
\begin{aligned}
(\alpha - \beta)\langle u, v\rangle &= \langle \alpha u, v\rangle - \langle u, \bar{\beta} v\rangle \\
&= \langle Tu, v\rangle - \langle u, T^* v\rangle \\
&= 0
\end{aligned}
$$

Proofs

Prop'n 7.22:

The eigenvectors of a normal operator of distinct eigenvalues are orthogonal.

Proof.

S'pose α, β are distinct eigenvalues, $Tu = \alpha u$ and $Tv = \beta v$.

$$\begin{aligned}(\alpha - \beta)\langle u, v\rangle &= \langle \alpha u, v\rangle - \langle u, \bar{\beta} v\rangle \\ &= \langle Tu, v\rangle - \langle u, T^* v\rangle \\ &= 0\end{aligned}$$

So, $\langle u, v\rangle = 0$.

References

[Axl14] Sheldon Axler.
Linear Algebra Done Right.
Undergraduate Texts in Mathematics. Springer Cham, 2014.