Lecture 19: The Spectral Theorem(s)

MATH 110-3

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Recall: Definitions

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Def'n:

S'pose $T \in \mathcal{L}(V, W)$. The **adjoint** of T is the function $T^* : W \to V$ such that

$$\langle Tv, w \rangle = \langle v, T^* w \rangle$$

for every $v \in V$ and $w \in W$.

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Def'n:

An operator T on an inner product space is called **normal** if $TT^* = T^*T$.

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Condition for normality:

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Norm of an orthonormal linear combo:

If $e_1, \ldots, e_m$ is an orthonormal list of vectors in V, then $||a_1e_1 + \ldots + a_me_m||^2 = |a_1|^2 + \ldots + |a_m|^2$ for any $a_i \in \mathbb{F}$.

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Schur's Theorem:

S'pose V is a finite dimensional $\mathbb{C}$-vector space. Then $T \in \mathcal{L}(V)$ has an upper triangular matrix with respect to some orthonormal basis.

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S'pose V is a finite dimensional $\mathbb{C}$-vector space. Then $T \in \mathcal{L}(V)$ has an upper triangular matrix with respect to some orthonormal basis.

Cauchy-Schwarz:

Suppose $u, v \in V$. Then $|\langle u, v\rangle| \leq ||u||||v||$. Equality is reached if and only if one of u or v is a scalar multiple of the other.

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Goal: Characterize when an operator has a diagonal matrix with respect to an orthonormal basis.

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- Case 1: $\mathbb{F} = \mathbb{C}$
- Case 2: $\mathbb{F} = \mathbb{R}$

The Complex Spectral Theorem

Theorem:

Suppose $\mathbb{F} = \mathbb{C}$ and $T = \mathcal{L}(V)$. The following are equivalent:

1. T is normal
2. V has an orthonormal basis consisting of eigenvectors of T
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Proof.

We already know (2) $\Leftrightarrow$ (3) by conditions for diagonalizability. We show (1) $\Leftrightarrow$ (3).

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Suppose T has a diagonal matrix with respect to some orthogonal basis.

Then T^* is the conjugate transpose and also diagonal.

Diagonal matrices commute. Thus, T is normal.

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Proof (cont'd).

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Proof (cont'd).

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Schur's Theorem, tells us there is an orthonormal basis $e_1, \ldots, e_n$ of V where T has an upper triangular matrix:

$$\mathcal{M}(T, e_1, \ldots, e_n) = \begin{pmatrix} a_{1,1} & \ldots & a_{1,n} \\ & \ldots & \vdots \\ 0 & & a_{n,n} \end{pmatrix}$$

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Proof (cont'd).

On the other hand, suppose T is normal.

Schur's Theorem, tells us there is an orthonormal basis $e_1, \ldots, e_n$ of V where T has an upper triangular matrix:

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We will show this is actually diagonal.

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We have

$$||Te_1||^2 = |a_{1,1}|^2$$

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since T is normal, these are equal.

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since T is normal, these are equal.

Thus, all the entries $a_{1,k}$ are zero except possibly $a_{1,1}$.

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Proof (cont'd).

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Thus, all the entries $a_{2,k}$ are zero except possibly $a_{2,2}$.
Repeating, we see that $\mathcal{M}(T)$ is **diagonal**!

The Real Spectral Theorem

Theorem:

Suppose $\mathbb{F} = \mathbb{R}$ and $T \in \mathcal{L}(V)$. Then the following are equivalent:

1. T is self-adjoint
2. V has an orthonormal basis consiting of eigenvectors of T
3. T has a diagonal matrix with respect to some orthonormal basis of V

Lemma's

Prop'n:

S'pose $T \in \mathcal{L}(V)$ is self-adjoint and $b, c \in \mathbb{R}$ such that $b^2 < 4c$. Then $T^2 + bT + cI$ is invertible.

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$$\begin{aligned}
\langle (T^2 + bT + cI)v, v\rangle &= \langle T^2 v, v\rangle + b\langle Tv, v\rangle + c\langle v, v\rangle \\
&= \langle Tv, Tv\rangle + b\langle Tv, v\rangle + c||v||^2 \\
&\geq ||Tv||^2 - |b|||Tv||||v|| + c||v||^2 \\
&= \left(||Tv|| - \frac{|b|||v||^2}{2}\right)^2 + \left(c - \frac{b^2}{4}\right)||v||^2 \\
&> 0
\end{aligned}$$

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\end{aligned}$$

Implies $(T^2 + bT + cI)v \neq 0$ (bc self-adjoint) and $(T^2 + bT + cI)$ is injective.

Reminder

Notice that we used, from last lecture:

Prop'n [Axl14]:

For T a self-adjoint operator on V a $\mathbb{C}$ or $\mathbb{R}$-vector space, such that $\langle Tv, v\rangle = 0$ for all $v \in V$. Then $T = 0$.

In general, if V is a $\mathbb{R}$-vector space if T is not self-adjoint $\langle Tv, v\rangle = 0$ for all $v \in V$ does not imply $T = 0$.

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$$0 = a_0v + a_1Tv + \ldots + a_nT^nv$$

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- Can factorize as:

$$c(T^2 + b_1T + c_1I)\cdots(T^2 + b_MT + c_MI)(T - \lambda_1I)\cdots(T - \lambda_mI) = 0$$

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- But the quadratic terms are invertible, so for some λ_j, $T - \lambda_j I$ is not injective and T has an eigenvalue.

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S'pose $T \in \mathcal{L}(V)$ is self-adjoint and U is a subspace of V that is invariant under T. Then

1. $U^\perp$ is invariant under T
2. $T|_U \in \mathcal{L}(U)$ is self-adjoint
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Implies $Tv \in U^\perp$.

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Implies $T|_U$ is self-adjoint.

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For (3), replace U with $U^\perp$ in (2). $\square$.

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Theorem:

Suppose $\mathbb{F} = \mathbb{R}$ and $T \in \mathcal{L}(V)$. Then the following are equivalent:

1. T is self-adjoint
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3. T has a diagonal matrix with respect to some orthonormal basis of V

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Proof. We show (3) $\implies$ (1),i.e. desired diagonal matrix implies self-adjoint.

A diagonal matrix is equal to its transpose and the complex conjugate of any real number is itself, $T = T^*$.

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Proof (cont'd).

We (1) $\Leftrightarrow$ (2): self-adjoint gives basis of orthonormal eigenvectors, using induction on dimension.

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Base case:

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Base case: $\dim V = 1$. Pick the basis to be $\{1\}$

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Induction: Assume $T \in \mathcal{L}(V)$ self-adjoint and all vector spaces of smaller dimension have orthonormal eigenbases.

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Proof (cont'd).

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We know T has an eigenvalue λ.

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Proof (cont'd).

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We know T has an eigenvalue λ.
Choose u such that $Tu = \lambda u$ and $||u|| = 1$.

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Proof (cont'd).

Induction: Assume $T \in \mathcal{L}(V)$ self-adjoint and all vector spaces of smaller dimension have orthonormal eigenbases.

We know T has an eigenvalue λ.
Choose u such that $Tu = \lambda u$ and $||u|| = 1$.

Let $U = \text{span}(u)$. U is a 1-dimensional invariant subspace
$T|_{U^\perp} \in \mathcal{L}(U^\perp)$ is self-adjoint.

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Proof (cont'd).

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By hypothesis, $T|_{U^\perp}$ has an orthonormal basis of eigenvectors.

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By hypothesis, $T|_{U^\perp}$ has an orthonormal basis of eigenvectors.

Adjoin this basis to u, found a basis of orthonormal eigenvectors of V.

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Proof (cont'd).

So we've show (3) $\implies$ (1) $\implies$ (2).

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The Real Spectral Theorem

Proof (cont'd).

So we've show (3) $\implies$ (1) $\implies$ (2). And we already know, (2) $\implies$ (3). So, we're done! $\square$.

References

[Axl14] Sheldon Axler.
Linear Algebra Done Right.
Undergraduate Texts in Mathematics. Springer Cham, 2014.