Lecture 1: Introduction and Vector Spaces

MATH 110-3

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Introductions

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You:

- Name
- Year at Berkeley or elsewhere
- A hobby

Logistics

- Syllabus
- Textbook
- Assignments and Exams
- LaTeX
- Gradescope
- Website
- *First Homework

Vector Spaces

Vector Spaces

Loosely, we care about

Objects:

$$\text{vector} \in \text{Vector Spaces}$$

$$\text{scalars} \in \text{Fields}$$

Fields

Def'n:

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- commutativity
- associativity
- identities
- additive inverses
- multiplicative inverse
- distributive property

(**Axler 1.3**)

Field Examples

1.3 Properties of complex arithmetic

commutativity
$\alpha + \beta = \beta + \alpha$ and $\alpha\beta = \beta\alpha$ for all $\alpha, \beta \in \mathbf{C}$;

associativity
$(\alpha+\beta)+\lambda = \alpha+(\beta+\lambda)$ and $(\alpha\beta)\lambda = \alpha(\beta\lambda)$ for all $\alpha, \beta, \lambda \in \mathbf{C}$;

identities
$\lambda + 0 = \lambda$ and $\lambda 1 = \lambda$ for all $\lambda \in \mathbf{C}$;

additive inverse
for every $\alpha \in \mathbf{C}$, there exists a unique $\beta \in \mathbf{C}$ such that $\alpha + \beta = 0$;

multiplicative inverse
for every $\alpha \in \mathbf{C}$ with $\alpha \neq 0$, there exists a unique $\beta \in \mathbf{C}$ such that $\alpha\beta = 1$;

distributive property
$\lambda(\alpha + \beta) = \lambda\alpha + \lambda\beta$ for all $\lambda, \alpha, \beta \in \mathbf{C}$.

- $\mathbb{R}$

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- $\mathbb{C}$
- $\mathbb{Z}$? Non-example
- $\{0, 1\}$? Yes, $\mathbb{F}_2$

Vector Spaces

Def'n:

An $\mathbb{F}$-**Vector Space**, V, is a collection of objects called ***vectors*** with two operations:

- addition:

$$\vec{u}, \vec{v} \in V$$

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…satisfying properties in Axler 1.9.

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…satisfying properties in Axler 1.9.

We can**not** multiply $\vec{u}$ and $\vec{v}$.

Vector Space Examples

1.19 Definition ***vector space***

A ***vector space*** is a set V along with an addition on V and a scalar multiplication on V such that the following properties hold:

commutativity
$u + v = v + u$ for all $u, v \in V$;

associativity
$(u + v) + w = u + (v + w)$ and $(ab)v = a(bv)$ for all $u, v, w \in V$ and all $a, b \in \mathbf{F}$;

additive identity
there exists an element $0 \in V$ such that $v + 0 = v$ for all $v \in V$;

additive inverse
for every $v \in V$, there exists $w \in V$ such that $v + w = 0$;

multiplicative identity
$1v = v$ for all $v \in V$;

distributive properties
$a(u + v) = au + av$ and $(a + b)v = av + bv$ for all $a, b \in \mathbf{F}$ and all $u, v \in V$.

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- $\mathbb{F}^\infty$?

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- $\mathbb{R}^n$
- $\mathbb{F}^n$
- $\mathbb{F}^\infty$?
- $\mathbb{F}^S$ i.e. $\mathbb{R}^{[0,1]}$? (set of functions $S \to \mathbb{F}$)

Propositions

Prop'n 1:

A vector space has a unique additive identity.

Prop'n 2:

Each element (vector) in a vector space has a unique additive inverse.

Propositions

Proof of Prop'n 1:

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Let V be a vector space.

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Let $\vec{0}, \vec{0}^*$ both be additive identities.

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Let V be a vector space.

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Then $\vec{0} = \vec{0} + \vec{0}^*$ because $\vec{0}^*$ is an additive identity (Def'n).

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Proof of Prop'n 1:

Let V be a vector space.

Let $\vec{0}, \vec{0}^*$ both be additive identities.

Then $\vec{0} = \vec{0} + \vec{0}^*$ because $\vec{0}^*$ is an additive identity (Def'n).

And $\vec{0} + \vec{0}^* = \vec{0}^*$ because $\vec{0}$ is an additive identity (Def'n).

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Thus, $\vec{0} = \vec{0} + \vec{0}^* = \vec{0}^*$.

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Thus, $\vec{0} = \vec{0} + \vec{0}^* = \vec{0}^*$.

Q.E.D.

$\square$

Propositions

Can you do 2?

Prop'n 2:

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Solution: Axler 1.26

References

[Axl14] Sheldon Axler.
Linear Algebra Done Right.
Undergraduate Texts in Mathematics. Springer Cham, 2014.