

Lecture 1: Introduction and Vector Spaces

MATH 110-3

Franny Dean

June 20, 2023

Introductions

Franny Dean, *she/her* frances_dean@berkeley.edu

You:

- Name
- Year at Berkeley or elsewhere
- A hobby

Logistics

- Syllabus
- Textbook
- Assignments and Exams
- LaTeX
- Gradescope
- Website
- *First Homework

Lecture 1

Vector Spaces



Loosely, we care about

Objects:

vector \in Vector Spaces

 $\text{scalars} \in \text{Fields}$

FD • MATH 110 • June 20, 2023

Fields

Def'n:

A **field** is a collection of objects with two binary operations (adding and multiplying), special elements, 0 and 1, $0 \neq 1$ satisfying nice properties:

Fields

Def'n:

A **field** is a collection of objects with two binary operations (adding and multiplying), special elements, 0 and 1, $0 \neq 1$ satisfying nice properties:

- commutativity
- associativity
- identities
- additive inverses
- multiplicative inverse
- distributive property

(Axler 1.3)

1.3 Properties of complex arithmetic

commutativity

 $\alpha + \beta = \beta + \alpha$ and $\alpha\beta = \beta\alpha$ for all $\alpha, \beta \in \mathbb{C}$;

associativity

 $(\alpha + \beta) + \lambda = \alpha + (\beta + \lambda)$ and $(\alpha\beta)\lambda = \alpha(\beta\lambda)$ for all $\alpha, \beta, \lambda \in \mathbb{C}$;

identities

 $\lambda + 0 = \lambda$ and $\lambda 1 = \lambda$ for all $\lambda \in \mathbb{C}$;

additive inverse

for every $\alpha \in C$, there exists a unique $\beta \in C$ such that $\alpha + \beta = 0$;

multiplicative inverse

for every $\alpha \in C$ with $\alpha \neq 0$, there exists a unique $\beta \in C$ such that $\alpha\beta = 1$;

distributive property



1.3 Properties of complex arithmetic

commutativity

 $\alpha + \beta = \beta + \alpha$ and $\alpha\beta = \beta\alpha$ for all $\alpha, \beta \in \mathbb{C}$;

associativity

 $(\alpha + \beta) + \lambda = \alpha + (\beta + \lambda)$ and $(\alpha\beta)\lambda = \alpha(\beta\lambda)$ for all $\alpha, \beta, \lambda \in \mathbb{C}$;

identities

 $\lambda + 0 = \lambda$ and $\lambda 1 = \lambda$ for all $\lambda \in \mathbb{C}$;

additive inverse

for every $\alpha \in C$, there exists a unique $\beta \in C$ such that $\alpha + \beta = 0$;

multiplicative inverse

for every $\alpha \in C$ with $\alpha \neq 0$, there exists a unique $\beta \in C$ such that $\alpha\beta = 1$;

distributive property

 $\lambda(\alpha + \beta) = \lambda\alpha + \lambda\beta$ for all $\lambda, \alpha, \beta \in \mathbb{C}$.



(())

1.3 Properties of complex arithmetic

commutativity

 $\alpha + \beta = \beta + \alpha$ and $\alpha\beta = \beta\alpha$ for all $\alpha, \beta \in \mathbb{C}$;

associativity

 $(\alpha + \beta) + \lambda = \alpha + (\beta + \lambda)$ and $(\alpha\beta)\lambda = \alpha(\beta\lambda)$ for all $\alpha, \beta, \lambda \in \mathbb{C}$;

identities

 $\lambda + 0 = \lambda$ and $\lambda 1 = \lambda$ for all $\lambda \in \mathbb{C}$;

additive inverse

for every $\alpha \in C$, there exists a unique $\beta \in C$ such that $\alpha + \beta = 0$;

multiplicative inverse

for every $\alpha \in C$ with $\alpha \neq 0$, there exists a unique $\beta \in C$ such that $\alpha\beta = 1$;

distributive property

 $\lambda(\alpha + \beta) = \lambda\alpha + \lambda\beta$ for all $\lambda, \alpha, \beta \in \mathbb{C}$.



1.3 Properties of complex arithmetic

commutativity

 $\alpha + \beta = \beta + \alpha$ and $\alpha\beta = \beta\alpha$ for all $\alpha, \beta \in \mathbb{C}$;

associativity

 $(\alpha + \beta) + \lambda = \alpha + (\beta + \lambda)$ and $(\alpha\beta)\lambda = \alpha(\beta\lambda)$ for all $\alpha, \beta, \lambda \in \mathbb{C}$;

identities

 $\lambda + 0 = \lambda$ and $\lambda 1 = \lambda$ for all $\lambda \in \mathbb{C}$;

additive inverse

for every $\alpha \in C$, there exists a unique $\beta \in C$ such that $\alpha + \beta = 0$;

multiplicative inverse

for every $\alpha \in C$ with $\alpha \neq 0$, there exists a unique $\beta \in C$ such that $\alpha\beta = 1$;

distributive property



1.3 Properties of complex arithmetic

commutativity

 $\alpha + \beta = \beta + \alpha$ and $\alpha\beta = \beta\alpha$ for all $\alpha, \beta \in \mathbb{C}$;

associativity

 $(\alpha + \beta) + \lambda = \alpha + (\beta + \lambda)$ and $(\alpha\beta)\lambda = \alpha(\beta\lambda)$ for all $\alpha, \beta, \lambda \in \mathbb{C}$;

identities

 $\lambda + 0 = \lambda$ and $\lambda 1 = \lambda$ for all $\lambda \in \mathbb{C}$;

additive inverse

for every $\alpha \in C$, there exists a unique $\beta \in C$ such that $\alpha + \beta = 0$;

multiplicative inverse

for every $\alpha \in C$ with $\alpha \neq 0$, there exists a unique $\beta \in C$ such that $\alpha\beta = 1$;

distributive property





1.3 Properties of complex arithmetic

commutativity

 $\alpha + \beta = \beta + \alpha$ and $\alpha\beta = \beta\alpha$ for all $\alpha, \beta \in \mathbb{C}$;

associativity $(\alpha + \beta) + \lambda = \alpha + (\beta + \lambda) \text{ and } (\alpha\beta)\lambda = \alpha(\beta\lambda) \text{ for all } \alpha, \beta, \lambda \in \mathbb{C};$

identities

 $\lambda + 0 = \lambda$ and $\lambda 1 = \lambda$ for all $\lambda \in \mathbb{C}$;

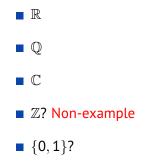
additive inverse

for every $\alpha \in C$, there exists a unique $\beta \in C$ such that $\alpha + \beta = 0$;

multiplicative inverse

for every $\alpha \in C$ with $\alpha \neq 0$, there exists a unique $\beta \in C$ such that $\alpha\beta = 1$;

distributive property



1.3 Properties of complex arithmetic

commutativity

 $\alpha + \beta = \beta + \alpha$ and $\alpha\beta = \beta\alpha$ for all $\alpha, \beta \in \mathbb{C}$;

associativity $(\alpha + \beta) + \lambda = \alpha + (\beta + \lambda)$ and $(\alpha\beta)\lambda = \alpha(\beta\lambda)$ for all $\alpha, \beta, \lambda \in \mathbb{C}$;

identities

 $\lambda + 0 = \lambda$ and $\lambda 1 = \lambda$ for all $\lambda \in \mathbb{C}$;

additive inverse

for every $\alpha \in C$, there exists a unique $\beta \in C$ such that $\alpha + \beta = 0$;

multiplicative inverse

for every $\alpha \in C$ with $\alpha \neq 0$, there exists a unique $\beta \in C$ such that $\alpha\beta = 1$;

distributive property

 $\lambda(\alpha + \beta) = \lambda\alpha + \lambda\beta$ for all $\lambda, \alpha, \beta \in \mathbb{C}$.

Q
C
Z? Non-example

 \mathbb{R}

■ {0,1}? Yes, 𝔽₂

Vector Spaces

Def'n:

An \mathbb{F} -Vector Space, *V*, is a collection of objects called *vectors* with two operations:

addition:

 $\vec{u}, \vec{v} \in V$ $\vec{u} + \vec{v} \in V$

Vector Spaces

Def'n:

An \mathbb{F} -Vector Space, *V*, is a collection of objects called *vectors* with two operations:

addition:

$$ec{u}, ec{v} \in V$$

 $ec{u} + ec{v} \in V$

scalar multiplication:

$$\lambda \in \mathbb{F}, \vec{v} \in V$$
$$\lambda \vec{v} \in V$$

... satisfying properties in Axler 1.9.

Vector Spaces

Def'n:

An \mathbb{F} -Vector Space, *V*, is a collection of objects called *vectors* with two operations:

addition:

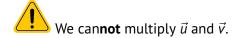
$$\vec{u}, \vec{v} \in V$$

 $\vec{u} + \vec{v} \in V$

scalar multiplication:

$$\lambda \in \mathbb{F}, \vec{v} \in V$$
$$\lambda \vec{v} \in V$$

... satisfying properties in Axler 1.9.



1.19 Definition vector space

A vector space is a set V along with an addition on V and a scalar multiplication on V such that the following properties hold:

commutativity

u + v = v + u for all $u, v \in V$;

associativity

(u + v) + w = u + (v + w) and (ab)v = a(bv) for all $u, v, w \in V$ and all $a, b \in \mathbf{F}$;

additive identity

there exists an element $0 \in V$ such that v + 0 = v for all $v \in V$;

additive inverse

for every $v \in V$, there exists $w \in V$ such that v + w = 0;

multiplicative identity

1v = v for all $v \in V$;

distributive properties

a(u+v) = au + av and (a+b)v = av + bv for all $a, b \in \mathbf{F}$ and all $u, v \in V$.



1.19 Definition vector space

A vector space is a set V along with an addition on V and a scalar multiplication on V such that the following properties hold:

commutativity

u + v = v + u for all $u, v \in V$;

associativity

(u + v) + w = u + (v + w) and (ab)v = a(bv) for all $u, v, w \in V$ and all $a, b \in \mathbf{F}$;

additive identity

there exists an element $0 \in V$ such that v + 0 = v for all $v \in V$;

additive inverse

for every $v \in V$, there exists $w \in V$ such that v + w = 0;

multiplicative identity

1v = v for all $v \in V$;

distributive properties

a(u+v) = au + av and (a+b)v = av + bv for all $a, b \in \mathbf{F}$ and all $u, v \in V$.



R

1.19 Definition vector space

A vector space is a set V along with an addition on V and a scalar multiplication on V such that the following properties hold:

commutativity

u + v = v + u for all $u, v \in V$;

associativity

(u + v) + w = u + (v + w) and (ab)v = a(bv) for all $u, v, w \in V$ and all $a, b \in \mathbf{F}$;

additive identity

there exists an element $0 \in V$ such that v + 0 = v for all $v \in V$;

additive inverse

for every $v \in V$, there exists $w \in V$ such that v + w = 0;

multiplicative identity

1v = v for all $v \in V$;

distributive properties

a(u + v) = au + av and (a + b)v = av + bv for all $a, b \in \mathbf{F}$ and all $u, v \in V$.



1.19 Definition vector space

A vector space is a set V along with an addition on V and a scalar multiplication on V such that the following properties hold:

commutativity

u + v = v + u for all $u, v \in V$;

associativity

(u + v) + w = u + (v + w) and (ab)v = a(bv) for all $u, v, w \in V$ and all $a, b \in \mathbf{F}$;

additive identity

there exists an element $0 \in V$ such that v + 0 = v for all $v \in V$;

additive inverse

for every $v \in V$, there exists $w \in V$ such that v + w = 0;

multiplicative identity

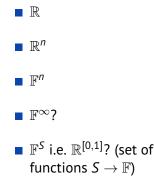
1v = v for all $v \in V$;

distributive properties

a(u + v) = au + av and (a + b)v = av + bv for all $a, b \in \mathbf{F}$ and all $u, v \in V$.



1.19 Definition vector space A vector space is a set V along with an addition on V and a scalar multiplication on V such that the following properties hold: commutativity u + v = v + u for all $u, v \in V$; associativity (u + v) + w = u + (v + w) and (ab)v = a(bv) for all $u, v, w \in V$ and all $a, b \in \mathbf{F}$: additive identity there exists an element $0 \in V$ such that v + 0 = v for all $v \in V$; additive inverse for every $v \in V$, there exists $w \in V$ such that v + w = 0; multiplicative identity 1v = v for all $v \in V$; distributive properties a(u + v) = au + av and (a + b)v = av + bv for all $a, b \in \mathbf{F}$ and all $u, v \in V$.



Prop'n 1:

A vector space has a unique additive identity.

Prop'n 2:

Each element (vector) in a vector space has a unique additive inverse.

Proof of Prop'n 1:

Lecture 1

Propositions

Proof of Prop'n 1:

Let *V* be a vector space.

Proof of Prop'n 1:

Let V be a vector space. Let $\vec{0}, \vec{0}^*$ both be additive identities.

Proof of Prop'n 1:

Let V be a vector space. Let $\vec{0}, \vec{0}^*$ both be additive identities. Then $\vec{0} = \vec{0} + \vec{0}^*$ because $\vec{0}^*$ is an additive identity (Def'n).

Proof of Prop'n 1:

Let \vec{V} be a vector space. Let $\vec{0}, \vec{0}^*$ both be additive identities. Then $\vec{0} = \vec{0} + \vec{0}^*$ because $\vec{0}^*$ is an additive identity (Def'n). And $\vec{0} + \vec{0}^* = \vec{0}^*$ because $\vec{0}$ is an additive identity (Def'n).

Proof of Prop'n 1:

Let V be a vector space. Let $\vec{0}, \vec{0}^*$ both be additive identities. Then $\vec{0} = \vec{0} + \vec{0}^*$ because $\vec{0}^*$ is an additive identity (Def'n). And $\vec{0} + \vec{0}^* = \vec{0}^*$ because $\vec{0}$ is an additive identity (Def'n). Thus, $\vec{0} = \vec{0} + \vec{0}^* = \vec{0}^*$.

Proof of Prop'n 1:

Let \vec{V} be a vector space. Let $\vec{0}, \vec{0}^*$ both be additive identities. Then $\vec{0} = \vec{0} + \vec{0}^*$ because $\vec{0}^*$ is an additive identity (Def'n). And $\vec{0} + \vec{0}^* = \vec{0}^*$ because $\vec{0}$ is an additive identity (Def'n). Thus, $\vec{0} = \vec{0} + \vec{0}^* = \vec{0}^*$. Q.E.D. Lecture 1

Propositions

Can you do 2?

Prop'n 2:

Each element (vector) in a vector space has a unique additive inverse.

Solution: Axler 1.26

Lecture 1



[Axl14] Sheldon Axler. Linear Algebra Done Right. Undergraduate Texts in Mathematics. Springer Cham, 2014.