Lecture 20: Positive Operators and Isometries

MATH 110-3

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Definition

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An operator $T \in \mathcal{L}(V)$ is called **positive** if T is self-adjoint and

$$\langle Tv, v \rangle \geq 0$$

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Examples:

- $U \subseteq V$, P_U is a positive operator
- $T \in \mathcal{L}(V)$ self-adjoint $b, c \in \mathbb{R}$ such that $b^2 < 4c$, we saw last lecture that $T^2 + bT + cI$ is a positive operator

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- Positive operator refers to $T \in \mathcal{L}(V)$. Positive-definite refers to $\mathcal{M}(T)$.
- A **positive semi-definite** matrix is one such that corresponds to an operator T such that $\langle Tv, v \rangle \geq 0$.
- The usual definition is a matrix such that $v^* \mathcal{M}(T) v \geq 0$. Why are these the same?

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Example:

$T \in \mathcal{L}(\mathbb{F}^3)$ defined $T(z_1, z_2, z_3) = (z_3, 0, 0)$

Then $R \in \mathcal{L}(\mathbb{F}^3)$ defined $R(z_1, z_2, z_3) = (z_2, z_3, 0)$ is a square root of T.

Characterization of Positive Operators

Prop'n:

For $T \in \mathcal{L}(V)$. The following are equivalent:

(a) T is positive
(b) T is self-adjoint and all the eigenvalues of T are nonnegative
(c) T has a positive square root
(d) T has a self-adjoint square root
(e) there exists an operator $R \in \mathcal{L}(V)$ such that $R^*R = T$

Proofs

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Then $T^* = (R^*R)^* = R^*R = T$. So T is self-adjoint.

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S'pose (d) : $T = R^2$ for self-adjoint R. Then $T = R^*R$ because $R = R^*$.

S'pose (e) : Let $R \in \mathcal{L}(V)$ such that $T = R^*R$.
Then $T^* = (R^*R)^* = R^*R = T$. So T is self-adjoint.
Also, for every $v \in V$,

$$\langle Tv, v\rangle = \langle R^*Rv, v\rangle = \langle Rv, Rv\rangle \geq 0.$$

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Let R be a positive square root.

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This determines R because we can pick a basis of eigenvectors by the Spectral Thm.

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Because R is positive, its eigenvalues are nonnegative and we can write them as $\sqrt{\lambda_i}$ for some λ_i.

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$$Rv = R(a_1 e_1 + \ldots + a_n e_n) = a_1 \sqrt{\lambda_1} e_1 + \ldots + a_n \sqrt{\lambda_n} e_n$$

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But $R^2v = Tv = \lambda v$.

$$a_1\lambda e_1 + \ldots + a_n\lambda e_n = a_1\lambda_1e_1 + \ldots + a_n\lambda_ne_n$$

and $a_j(\lambda - \lambda_j) = 0$ for $j = 1, \ldots, n$.

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$$R^2 v = a_1\lambda_1 e_1 + \ldots + a_n\lambda_n e_n.$$

But $R^2 v = Tv = \lambda v$.

$$a_1\lambda e_1 + \ldots + a_n\lambda e_n = a_1\lambda_1 e_1 + \ldots + a_n\lambda_n e_n$$

and $a_j(\lambda - \lambda_j) = 0$ for $j = 1, \ldots, n$. Hence,

$$v = \sum_{\lambda_j = \lambda} a_j e_j \qquad Rv = \sum_{\lambda_j=\lambda} a_j\sqrt{\lambda}e_j = \sqrt{\lambda}v$$

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Def'n:

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$$v = \langle v, e_1 \rangle e_1 + \ldots + \langle v, e_n \rangle e_n$$

$$||v||^2 = |\langle v, e_1 \rangle|^2 + \ldots + |\langle v, e_n \rangle|^2$$

$$Sv = \langle v, e_1 \rangle Se_1 + \ldots + \langle v, e_n \rangle Se_n$$

$$\ldots$$

Other Words for Isometry

- On a real inner product space, isometry = **orthogonal operator**.
- On a complex inner product space, isometry = **unitary operator**.

Characterization of Isometries

Prop'n:

TFAE for operator $S \in \mathcal{L}(V)$:

(a) S is an isometry

(b) $\langle Su, Sv \rangle = \langle u, v \rangle$ for all $u, v \in V$

(c) $Se_1, \ldots, Se_n$ is orthonormal for every orthonormal list $e_1, \ldots, e_n$

(d) there exists an orthonormal basis $e_1, \ldots, e_n$ of V such that $Se_1, \ldots, Se_n$ is orthonormal

(e) $S^*S = I$

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(g) S^* is an isometry

(h) S is invertible and $S^{-1} = S^*$

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Recall from discussion:

$$\langle u, v \rangle = \frac{||u+v||^2 - ||u-v||^2}{4}$$

for a real inner product space.

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Proof isometry is equivalent to preserving inner products:

We also have

$$\langle u, v \rangle = \frac{||u+v||^2 - ||u-v||^2 + ||u+iv||^2 i - ||u-iv||^2 i}{4}$$

for a complex inner product space.

Proof of Claim.

$$\begin{aligned} ||u+v||^2 &= \langle u+v, u+v \rangle \\ &= ||u||^2 + \langle u, v \rangle + \langle v, u \rangle + ||v||^2 \end{aligned}$$

Proof of Claim.

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which together gives

$$||u+v||^2 - ||u-v||^2 + ||u+iv||^2 i - ||u-iv||^2 i = 4\langle u, v \rangle$$

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Now that we can write inner products in terms of norms, we have the equivalence of (a) and (b).

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Isometries when $\mathbb{F} = \mathbb{C}$

Prop'n:

Suppose V is a complex vector space and $S \in \mathcal{L}(V)$. Then the following are equivalent:

- S is an isometry
- There is an orthonormal basis consisting of eigenvectors of S whose corresponding eigenvalues have absolute value 1.

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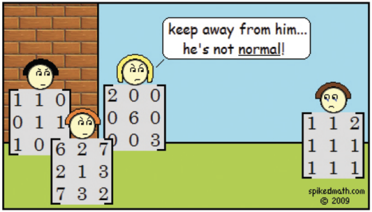
$$|\lambda_j| = |\lambda_j| ||e_j|| = ||\lambda_j e_j|| = ||Se_j|| = ||e_j|| = 1$$

Break

Discussion Questions

1. S'pose n is a positive integer. Define $T \in \mathcal{L}(\mathbb{F}^n)$ by $T(z_1, \ldots, z_n) = (0, z_1, \ldots, z_{n-1})$. Find a formula for $T^*(z_1, \ldots, z_n)$.
2. Let V be a $\mathbb{C}$-vector space and $T \in \mathcal{L}(V)$ be normal such that $T^9 = T^8$. Prove that the only possible eigenvalues of T are 0 and 1 and that $T^2 = T$.
3. Give an example of an operator $T \in \mathcal{L}(\mathbb{C}^4)$ that is normal but not self-adjoint.
4. Find the covered up entry.



Discussion Questions

5. Give a counter example to the following: If $T \in \mathcal{L}(V)$ is self-adjoint and there exists and orthonormal basis $e_1, \ldots, e_n$ of V such that $\langle Te_j, e_j \rangle \geq 0$ for each j, then T is a positive operator.
6. Prove that the sum of two positive operators on V is positive.
7. Suppose $T \in \mathcal{L}(V)$ is positve. Prove that T^k is positive for every positive integer k.
8. Give a counterexample: if $S \in \mathcal{L}(V)$ and there exists an orthonormal basis $e_1, \ldots, e_n$ of V such that $||Se_j|| = 1$ for each e_j, then S is an isometry.

Discussion Question Hints/Solutions

1. $T^*(z_1, \dots, z_n) = (z_2, \dots, z_n, 0)$
2. For the first part, use the Complex Spectral Thm to get a basis of orthonormal eigenvectors. $Te_j = \lambda_j e_j$. Applying T repeatedly we have $\lambda_j^9 = \lambda_j^8$. So $\lambda_j = 0$ or 1. Then applying T twice we see $T^2 e_j = \lambda_j^2 e_j = \lambda_j e_j = Te_j$.
3. $T = iI$ and $T^* = -iI$ works.
4. Its 1.

Discussion Question Hints/Solutions

5. One example: $V = \mathbb{F}^2$ with standard inner product. $T(x, y) = (y, x)$. $\langle Te_j, e_j \rangle = 0$ for each of the standard basis vectors but $\langle T(1, -1), (1, -1) \rangle = -2$.
6. Sum of two self-adjoint operators is self-adjoint. Also, $\langle (S + T)v, v \rangle = \langle Sv, v \rangle + \langle Tv, v \rangle \geq 0$.
7. Even case: $T^k v, v\rangle = \langle T^{2m} v, v \rangle = \langle T^m v, T^m v \rangle \geq 0$
   Odd case: $T^k v, v\rangle = \langle T^{2m+1} v, v \rangle = \langle T(T^m v), T^m v \rangle \geq 0$
8. One example: Define $S \in \mathcal{L}(\mathbb{F}^2)$ by $S(w, z) = (w + z, 0)$ with usual inner product. The standard basis is orthonromal and satisfies the condition. But $||S(1, -1)|| = 0$.

References

[Axl14] Sheldon Axler.
Linear Algebra Done Right.
Undergraduate Texts in Mathematics. Springer Cham, 2014.