Lecture 21: Generalized Eigenvectors

MATH 110-3

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- Recall $T^k = TT \cdots T$ (composition of operators k times)
- **Goal:** Decomposing/understanding more general operators

Null Spaces of Powers of an Operator

Prop'n:

S'pose $T \in \mathcal{L}(V)$. Then

$$\{0\} = \text{null } T^0 \subseteq \text{null } T^1 \subseteq \ldots \subseteq \text{null } T^k \subseteq \text{null } T^{k+1} \subseteq \ldots.$$

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Thus, $v \in \text{null } T^{k+1}$.

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$$T^k v \in \operatorname{null} T^{m+1} = \operatorname{null} T^m \implies v \in \operatorname{null} T^{m+k}$$

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But, dimension must grow in this chain, of subsets of V, contradiction.

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But we do have null $T^3 \oplus$ range $T^3 = V$ since
$T^3(x, y, z) = (0, 0, 125z)$ and

$$\text{null } T^3 = \{(x, y, 0) : x, y \in \mathbb{F}\}$$

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Generalized Eigenvectors

Motivation:

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Problem: Eigenvalues, upper triangular/diagonal matrices not enough

Solution: Generalized eigenvectors.

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In general, we do not have this decomposition.

Generalized Eigenvectors

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Remark. $E(\lambda, T) \subset G(\lambda, T)$

Description of Generalized Eigenspaces

Prop'n:

S'pose $T \in \mathcal{L}(V)$ and $\lambda \in \mathbb{F}$. Then $G(\lambda, T) = \text{null}\,(T - \lambda I)^{\dim V}$.

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Notice $\mathbb{C}^3 = G(0, T) \oplus G(5, T)$.

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Thus, $v \in \text{null}\,(T - \lambda I)^{\dim V}$.

Linearly Independent Generalized Eigenvectors

Prop'n [Axl14]:

Let $T \in \mathcal{L}(V)$. Suppose $\lambda_1, \ldots, \lambda_m$ are distinct eigenvalues of T and $v_1, \ldots, v_m$ are corresponding generalized eigenvectors. Then $v_1, \ldots, v_m$ are linearly independent.

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- The derivative operator on $\mathcal{P}_n(\mathbb{R})$

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Characterizations of Nilpotent Operators

Matrix of nilpotent operator:

S'pose N is a nilpotent operator on V. Then there is a basis of V for which the matrix $\mathcal{M}(N)$ has the form:

$$\begin{pmatrix} 0 & & * \\ & \cdots & \\ 0 & & 0 \end{pmatrix}$$

where entries on and below the diagonal are zero.

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- Same with rest of columns.

References

[Axl14] Sheldon Axler.
Linear Algebra Done Right.
Undergraduate Texts in Mathematics. Springer Cham, 2014.