Lecture 22: Decomposition of Operators

MATH 110-3

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July 31, 2023

S'pose $T \in \mathcal{L}(V)$ and $\lambda \in \mathbb{F}$:

Def'n:

We say $v \in V$ is a **generalized eigenvector** if there exists positive integer j such that $(T - \lambda I)^j v = 0$.

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$G(\lambda, T) = \text{null}\,(T - \lambda I)^{\dim V}$

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Prop'n 1:

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Prop'n 2:

Let $n = \dim V$. Then $V = \text{null}\, T^n \oplus \text{range}\, T^n$.

Main Result

Description of Operators on Complex Vector Spaces

S'pose V is a complex vector space and $T \in \mathcal{L}(V)$. Let $\lambda_1, \ldots, \lambda_m$ be the distinct eigenvalues of T. Then

1. $V = G(\lambda_1, T) \oplus \cdots \oplus G(\lambda_m, T)$;
2. each $G(\lambda_j, T)$ is invariant under T;
3. each $(T - \lambda_j I)|_{G(\lambda_j, T)}$ is nilpotent.

Description of Operators on Complex Vector Spaces

Proof.

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Proof. Let's start with 3!

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For 2:

Lemma:

S'pose $T \in \mathcal{L}(V)$ and $p \in \mathcal{P}(\mathbb{F})$. Then null $p(T)$ and range $p(T)$ are invariant under T.

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How can we use to prove 2?

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$$V = G(\lambda_1, T) \oplus \operatorname{range}(T - \lambda_1 I)^n.$$

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range $(T - \lambda_1 I)^n$ satisfies induction hypothesis.

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Call range $(T - \lambda_1 I)^n = U$.

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So $v_1 = 0$ and $v = u \in U$ and $v \in G(\lambda_k, T|_U)$. $\square$.

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Proof. Use previous result! □.

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- The **geometric multiplicity** of an eigenvalue λ of T is defined to be the dimension of the corresponding eigenspace $E(\lambda, T)$.

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$$\mathbb{C}^3 = G(6, T) \oplus G(7, T)$$

Sum of Multiplicities

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Proof. Use previous result! $\square$.

Multiplicity of Eigenvalues and Upper Triangular Matrices

Prop'n [Axl14]:

Suppose $T \in \mathcal{L}(V)$ and $\lambda \in \mathbb{F}$. Then for every basis of V with respect to which T has an upper triangular matrix, the number of times that λ appears on the diagonal of the matrix of T equals the algebraic multiplicity of λ as an eigenvalue of T.

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Proof: Homework challenge problem.

Block Diagonal Matrices

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Def'n:

A **block diagonal matrix** is a square matrix of the form

$$\begin{pmatrix} A_1 & & 0 \\ & \cdots & \\ 0 & & A_m \end{pmatrix}$$

where $A_1, \ldots, A_m$ are square matrices lying along the diagonal and all other entries are 0.

Example

8.28 **Example** The 5-by-5 matrix

$$A = \begin{pmatrix} \begin{pmatrix} 4 \end{pmatrix} & 0 \quad 0 & 0 \quad 0 \\ \begin{matrix} 0 \\ 0 \end{matrix} & \begin{pmatrix} 2 & -3 \\ 0 & 2 \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 \\ 0 \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{pmatrix} 1 & 7 \\ 0 & 1 \end{pmatrix} \end{pmatrix}$$

is a block diagonal matrix with

$$A = \begin{pmatrix} A_1 & & 0 \\ & A_2 & \\ 0 & & A_3 \end{pmatrix},$$

where

$$A_1 = \begin{pmatrix} 4 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 2 & -3 \\ 0 & 2 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 1 & 7 \\ 0 & 1 \end{pmatrix}.$$

Block Diagonal Matrix with Upper Triangular Blocks

Prop'n:

S'pose V is complex vector space and $T \in \mathcal{L}(V)$. Let $\lambda_1, \ldots, \lambda_m$ be the distinct eigenvalues of T, with multiplicities $d_1, \ldots, d_m$. Then there is a basis of V with respect to which T has a block diagonal matrix of the form

$$\begin{pmatrix} A_1 & & 0 \\ & \ldots & \\ 0 & & A_m \end{pmatrix}$$

where each A_j is a $d_j \times d_j$ upper triangular matrix of the form

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What is the matrix of $T|_{G(\lambda_j,T)} = (T - \lambda_j I)|_{G(\lambda_j,T)} + \lambda_j I|_{G(\lambda_j,T)}$?

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The desired form.

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What is the matrix with respect to this basis?

Blocks:

$$\begin{pmatrix} 6 & 3 \\ 0 & 6 \end{pmatrix}, (7)$$

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- Notice $T^3 = 0$.
- If $S^2 = T$, then $S^6 = T$.

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- So $T^2 = S^4 = SS^3 = 0$, but $T^2 \neq 0$, contradiction.

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$$\begin{aligned}
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- Just solve for a_i such that the RHS is $I + N$.

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- Let R_j be the root of $\lambda_j \left(I + \frac{N_j}{\lambda_j} \right)$.
- Then for $v = u_1 + \ldots + u_m$, $Rv = R_1 u_1 + \ldots + R_m u_m$ is the square root of T.

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