Lecture 24: Characteristic and Minimal Polynomials

MATH 110-3

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Definitions

Def'n:

Suppose V is a complex vector space and $T \in \mathcal{L}(V)$. Let $\lambda_1, \ldots, \lambda_m$ denote the distinct eigenvalues of T, with algebraic multiplicities $d_1, \ldots, d_m$. The polynomial

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The characteristic polynomial is $(z - 6)^2(z - 7)$.

Properties of the Characteristic Polynomial

Prop'n:

Suppose V is a complex vector space and $T \in \mathcal{L}(V)$. Then

1. the characteristic polynomial of T has degree $\dim V$;
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Suppose $T \in \mathcal{L}(V)$. Then the **minimal polynomial** of T is the unique monic polynomial of p of smallest degree such that $p(T) = 0$.

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- **Uniqueness**: S'pose q with same degree.
- $q(T) = 0$ implies $(p - q)(T) = 0$, contradiction! Why?

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The minimal polynomial is $z^5 - 6z + 3$.

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Prop'n:

Suppose $T \in \mathcal{L}(V)$ and $q \in \mathcal{P}(\mathbb{F})$. Then $q(T) = 0$ if and only if q is a multiple of the minimal polynomial of T.

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Corollary:

The characteristic polynomial is a polynomial multiple of the minimal polynomial.

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A computer can compute the roots to be
$-1.67, 0.51, 1.40, -0.12 + 1.59i, -0.12 - 1.59i$.

References

[Axl14] Sheldon Axler.
Linear Algebra Done Right.
Undergraduate Texts in Mathematics. Springer Cham, 2014.