Lectures 24: Jordan Form

MATH 110-3

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Today

We have...

S'pose V is complex vector space and $T \in \mathcal{L}(V)$. Let $\lambda_1, \ldots, \lambda_m$ be the distinct eigenvalues of T, with multiplicities $d_1, \ldots, d_m$. Then there is a basis of V with respect to which T has a block diagonal matrix of the form

$$\begin{pmatrix} A_1 & & 0 \\ & \ldots & \\ 0 & & A_m \end{pmatrix}$$

where each A_j is a $d_j \times d_j$ upper triangular matrix of the form

$$A_j = \begin{pmatrix} \lambda_j & & * \\ & \ldots & \\ 0 & & \lambda_j \end{pmatrix}.$$

But we can do even better!

Jordan Basis

Def'n:

Suppose $T \in \mathcal{L}(V)$. A basis is called a **Jordan basis** for T if with respect to this basis T has a block diagonal matrix of the form

$$\begin{pmatrix} A_1 & & 0 \\ & \ldots & \\ 0 & & A_p \end{pmatrix}$$

where each A_j is an upper triangular matrix of the form

$$A_j = \begin{pmatrix} \lambda_j & 1 & & 0 \\ & \ldots & \ldots & \\ & & \ldots & 1 \\ 0 & & & \lambda_j \end{pmatrix}.$$

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Prop'n:

Suppose V is a complex vector space. If $T \in \mathcal{L}(V)$, then there is a basis of V that is a Jordan basis for T.

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Suppose V is a complex vector space. If $T \in \mathcal{L}(V)$, then there is a basis of V that is a Jordan basis for T.

How do we prove this? Start with nilpotent operators again!

Bases for Nilpotent Operators

Example 1:

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The matrix is

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

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Example 2:

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There is no v such that $N^5v, N^4v, N^3v, N^2v, Nv, v$ forms a basis...

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Prop'n:

Suppose $N \in \mathcal{L}(V)$ is nilpotent. Then there exists vectors $v_1, \ldots, v_n \in V$ and nonnegative integers $m_1, \ldots, m_n$ such that

- $N^{m_1} v_1, \ldots, N v_1, v_1, \ldots, N^{m_n} v_n, \ldots, N v_n, v_n$ is a basis of V;
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Consider $N|_{\text{range } N} \in \mathcal{L}(\text{range } N)$. (*Why can we ignore range* $N = \{0\}$?)

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Proof (cont'd). Consider $N|_{\text{range } N} \in \mathcal{L}(\text{range } N)$...

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$$N^{m_1}v_1, \ldots, Nv_1, v_1, \ldots, N^{m_n}v_n, \ldots, Nv_n, v_n$$

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$$N^{m_1+1}u_1, \ldots, Nu_1, u_1, \ldots, N^{m_n+1}u_n, \ldots, Nu_n, u_n$$

We claim this list is linearly independent in V.

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- Apply N.

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- All coefficients except possibly those in front of $N^{m_i+1}u_i'$s are 0.

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We claim this list is linearly independent in V.

- Write out a linear combo equaling 0.
- Apply N.
- All coefficients except possibly those in front of $N^{m_i+1} u_i' s$ are 0.
- The $N^{m_i+1} u_i' s = N^{m_i} v_i$ are linearly independent.

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So we have

$$N^{m_1+1}u_1, \ldots, Nu_1, u_1, \ldots, N^{m_n+1}u_n, \ldots, Nu_n, u_n$$

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So we have

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Extend to a basis of V:

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Thus, we have a basis of the desired form. $\square$.

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Now for T not nilpotent, we have $V = G(\lambda_1, T) \oplus \ldots \oplus G(\lambda_m, T)$ where $(T - \lambda_j I)|_{G(\lambda_j, T)}$ are nilpotent.

Add the $\lambda_j's$ along the diagonal. Done. $\square$.

Example

Before we saw

$$\begin{pmatrix} (4) & 0 & 0 & 0 & 0 \\ 0 & \begin{pmatrix} 2 & -3 \\ 0 & 2 \end{pmatrix} & & 0 & 0 \\ 0 & & & 0 & 0 \\ 0 & 0 & 0 & \begin{pmatrix} 1 & 7 \\ 0 & 1 \end{pmatrix} & \\ 0 & 0 & 0 & & \end{pmatrix}$$

Example

Before we saw

$$\begin{pmatrix} (4) & 0 \quad 0 & 0 \quad 0 \\ 0 & \begin{pmatrix} 2 & -3 \\ 0 & 2 \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ 0 & & \\ 0 & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{pmatrix} 1 & 7 \\ 0 & 1 \end{pmatrix} \\ 0 & & \end{pmatrix}$$

Its Jordan form is

$$\begin{pmatrix} (4) & 0 \quad 0 & 0 \quad 0 \\ 0 & \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ 0 & & \\ 0 & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \\ 0 & & \end{pmatrix}$$

Jordan Form

We can actually say a bit more about what the Jordan form looks like:

- Each Jordan block has basis $(T - \lambda_j I)^{m_j} v_j, \ldots, (T - \lambda_j I)v_j, v_j$ where m_j is such that $(T - \lambda_j I)^{m_j+1} v_j = 0$.

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- There are as many λ_j as the ***algebraic multiplicity*** or the $\dim G(\lambda_j, T)$.
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Summary:

- There are as many λ_j as the ***algebraic multiplicity*** or the $\dim G(\lambda_j, T)$.
- The number of Jordan blocks is given by the eigenspace dimension or ***geometric multiplicity***.
- The size of the largest Jordan block is given by the power of $(z - \lambda_j)$ in the minimal polynomial.
  - $p(J_j) = 0 \Leftrightarrow (z - \lambda_j)^d | p(z)$ where d is the size of the Jordan block J_j.

Example

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What if the minimal polynomial was instead $(z-5)^2(z-3)(z-1)^2$?

Break

Discussion Questions

1. Suppose V is a complex vector space, $N \in \mathcal{L}(V)$ and 0 is the only eigenvalue of N. Prove that N is nilpotent.
2. Give an example of an operator T on a finite-dimensional *real* vector space such that 0 is the only eigenvalue of T but T is not nilpotent.
3. Suppose $T \in \mathcal{L}(\mathbb{C}^4)$ is such that the eigenvalues of T are $3, 4, 5$. Prove that $(T - 3I)^2(T - 5I)^2(T - 4I)^2 = 0$.
4. Find examples of operators on $\mathbb{C}^4$ such that
   4.1 The char poly is $(z-1)(z-5)^3$ and the min poly is $(z-1)(z-5)^2$.
   4.2 Both polys are $z(z-1)^2(z-3)$

Discussion Questions

5. Suppose $T \in \mathcal{L}(V)$. Prove that T is invertible if and only if the constant term in the minimal polynomial is nonzero.
6. Suppose $T \in \mathcal{L}(V)$ has minimal polynomial $4 + 5z - 6z^2 - 7z^3 + 2z^4 + z^5$. Find the minimal polynomial of T^{-1}.
7. Suppose V is an inner product space and $T \in \mathcal{L}(V)$. Suppose $a_0 + a_1 z + \ldots + a_{m-1} z^{m-1} + a_m z^m$ is the minimal polynomial of T. Prove that $\bar{a_0} + \bar{a_1} z + \ldots + \bar{a_{m-1}} z^{m-1} + \bar{a_m} z^m$ is the minimal polynomial of T^*.
8. What is the Jordan form for the operator on $\mathbb{C}^3$ given by $T(x, y, z) = (3x + 4y + 7z, 3y + 7z, 2z)$?

Discussion Question Hints/Solutions

1. Our description of operators on complex vector spaces says that $V = G(0, T) = \text{null}\,(T - 0I)^{\dim V} = \text{null}\, T^{\dim V}$. Thus, T is nilpotent.
2. $T(x, y, z) = (-y, x, 0)$ works.
3. The multiplicity of the eigenvalues must sum to 4. Characteristic polynomial is $(z-3)^a(z-4)^b(z-5)^c$ where $a+b+c=4$. This polynomial will divide the given polynomial. By Cayley-Hamilton, we have result.
4. Consider $T(z_1, z_2, z_3, z_4) = (z_1, 5z_2 + z_4, 5z_3, 5z_4)$ for the first one and $T(z_1, z_2, z_3, z_4) = (0, z_2 + z_3, z_3, 3z_4)$ for the second.

Discussion Question Hints/Solutions

5. T is invertible if and only if 0 is not an eigenvalue. Notice that this is equivalent to the constant term being nonzero in the minimal polynomial because its roots are the eigenvalues.
6. Multiply both sides by T^{-5} and divide by 4 to get $z^5 + 5/4z^4 - 3/2z^3 - 7/2z^2 + 1/4z + 1/4$.
7. Take the adjoint of both sides of the equation with T plugged in. If there were another polynomial of lower degree, we could do the same process and contradict our choice of minimal polynomial for T.
8. $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix}$

References

[Axl14] Sheldon Axler.
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