Lecture 25: Trace

MATH 110-3

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August 3, 2023

Definition

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- If $\mathbb{F} = \mathbb{C}$, then the **trace** of T is the sum of the eigenvalues of T, with each eigenvalue repeated according to its multiplicity.

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Suppose $T \in \mathcal{L}(V)$.

- If $\mathbb{F} = \mathbb{C}$, then the **trace** of T is the sum of the eigenvalues of T, with each eigenvalue repeated according to its multiplicity.
- (If $\mathbb{F} = \mathbb{R}$, then the **trace** of T is the sum of the eigenvalues of the operator considered over $\mathbb{C}$ repeated according to multiplicity.)

Example

$T \in \mathcal{L}(\mathbb{C}^3)$ is the operator whose matrix is

$$\begin{pmatrix} 3 & -1 & -2 \\ 3 & 2 & -3 \\ 1 & 2 & 0 \end{pmatrix}.$$

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$$(z-1)(z-(2+3i))(z-(2-3i)) = z^3 - 5z^2 + 17z - 13$$

Trace and Characteristic Polynomial

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Prop'n:

Suppose $T \in \mathcal{L}(V)$. Let $n = \dim V$. Then trace T equals the negative of the coefficient of z^{n-1} in the characteristic polynomial of T.

Computing the Trace

- For upper triangular matrix form for an operator, the trace can be computed using the sum of the diagonal entries.

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Let's show this!

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Recall:

If $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$, then $\mathcal{M}(ST) = \mathcal{M}(S)\mathcal{M}(T)$.

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We can restate this with respect to bases as...

The matrix of the product of linear maps:

Suppose $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ and $w_1, \ldots, w_n$ are all bases of V. Suppose $S, T \in \mathcal{L}(V)$. Then

$$\mathcal{M}(ST, (u_1, \ldots, u_n), (w_1, \ldots, w_n)) =$$

$$\mathcal{M}(S, (v_1, \ldots, v_n), (w_1, \ldots, w_n))\mathcal{M}(T, (u_1, \ldots, u_n), (v_1, \ldots, v_n)).$$

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Thus, in our notation, we have

Matrix of the identity with respect to two bases:

Suppose $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ are bases of V. Then the matrices $\mathcal{M}(I, (u_1, \ldots, u_n), (v_1, \ldots, v_n))$ and $\mathcal{M}(I, (v_1, \ldots, v_n), (u_1, \ldots, u_n))$ are invertible and are inverses of each other.

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Proof.

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Proof. "Plug in" to previous result:

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$$\mathcal{M}(I,((1,0),(0,1)),((4,2),(5,3))) = \begin{pmatrix} \frac{3}{2} & \frac{-5}{2} \\ -1 & 2 \end{pmatrix}$$

Change of Basis Formula

Prop'n:

Suppose $T \in \mathcal{L}(V)$. Let $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ be the bases of V. Let $A = \mathcal{M}(I, (u_1, \ldots, u_n), (v_1, \ldots, v_n))$. Then

$$\mathcal{M}(T, (u_1, \ldots, u_n)) = A^{-1}\mathcal{M}(T, (v_1, \ldots, v_n))A.$$

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Trace of a matrix of operator does not depend on basis

Prop'n:

Let $T \in \mathcal{L}(V)$. Suppose $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ are bases of V. Then

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Corollary

Suppose $T \in \mathcal{L}(V)$. Then trace T = trace $\mathcal{M}(T)$.

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S'pose $S, T \in \mathcal{L}(V)$. Then $\text{trace}\,(S + T) = \text{trace}\, S + \text{trace}\, T$.

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Proof. S'pose $S, T \in \mathcal{L}(V)$. Choose a basis of V. Then

$$\begin{aligned}
\operatorname{trace}(ST - TS) &= \operatorname{trace}(ST) - \operatorname{trace}(TS) \\
&= \operatorname{trace}\mathcal{M}(ST) - \operatorname{trace}\mathcal{M}(TS) \\
&= \operatorname{trace}\mathcal{M}(S)\mathcal{M}(T) - \operatorname{trace}\mathcal{M}(T)\mathcal{M}(S) \\
&= 0
\end{aligned}$$

More Applications

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If someone tells you that -48 and 24 are eigenvalues of T, how could you tell what the third eigenvalue was?

References

[Axl14] Sheldon Axler.
Linear Algebra Done Right.
Undergraduate Texts in Mathematics. Springer Cham, 2014.