Lecture 26: Determinant

MATH 110-3

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Determinant of an Operator

Def'n:

Suppose $T \in \mathcal{L}(V)$ and V is a $\mathbb{C}$ vector space. The **determinant** of T, denoted $\det T$, is the product of the eigenvalues of T each repeated according to its algebraic multiplicity.

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Then the eigenvalues are $1, 2+3i, 2-3i$ and the determinant is $\det T = 1 \cdot (2+3i) \cdot (2-3i) = 13$.

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Then the eigenvalues are $6, 2$ and the determinant is $\det T = 6 \cdot 2 \cdot 2 = 24$.

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or

$$z^n - (\text{trace } T) z^{n-1} + \ldots + (-1)^n (\det T).$$

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Thus, T is invertible if and only if the determinant is non-zero. $\square$.

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Then

$$\det(zI - T) = (z - \lambda_1) \cdots (z - \lambda_n)$$

repeated according to the multiplicities of the eigenvalues of T. □

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We need *permutations*!

Detour: Permutations

Def'n:

- A **permutation** of $(1, 2, \ldots, n)$ is a list $(m_1, m_2, \ldots, m_n)$ that contains each of the numbers $1, 2, \ldots, n$ exactly once.

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- 1×1 matrix: $\det A = A_{1,1}$
- 2×2 matrix: the two elements of perm 2 give

$$\det \begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{pmatrix} = A_{1,1}A_{2,2} - A_{2,1}A_{1,2}$$

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- Upper triangular matrix: only permutation with no entries below the diagonal is $(1, 2, \ldots, n)$ giving the diagonal.

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Proof. $\det A = -\det A$ implies $\det A = 0$.

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The number of steps being even corresponds to $\text{sign}\left((m_1, \ldots, m_n)\right) = 1$ and the number of steps being odd corresponds to $\text{sign}\left((m_1, \ldots, m_n)\right) = -1$.

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- Where the c_i are of the form:

$$\sum \text{sign (permutation)} A_{m_1,l} \cdots \widehat{A_{m_k,k}} \cdots A_{m_n,l}$$

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- Let e_k denote the $n \times 1$ matrix that equals 1 in the k^{th} entry and 0 elsewhere (standard basis vectors as matrices).
- Then $Ae_k = A_{\cdot,k}$ and $Be_k = B_{\cdot,k}$.
- And $B_{\cdot,k} = \sum_{m=1}^{n} B_{m,k} e_m$.

Determinant is Multiplicative

Prop'n:

Suppose A and B are same size square matrices. Then

$$\det(AB) = \det(BA) = (\det B)(\det A).$$

Proof. We will show $\det(AB) = (\det B)(\det A)$.

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- And $B_{\cdot,k} = \sum_{m=1}^{n} B_{m,k} e_m$.
- And recall $AB = (AB_{\cdot,1} \cdots AB_{\cdot,n})$.

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- $= \det(A) \cdot \det(B)$.

Determinant of an Operator is the Determinant of its Matrix

Prop'n:

For $T \in \mathcal{L}(V)$, $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ bases of V. Then $\det \mathcal{M}(T, (u_1, \ldots, u_n)) = \det \mathcal{M}(T, (v_1, \ldots, v_n))$.

Proof.

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Suppose $T \in \mathcal{L}(\mathbb{R}^n)$ is a positive operator and $\Omega \subset \mathbb{R}^n$. Then volume $T(\Omega) = (\det T)(\text{volume } \Omega)$.

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- Then the box has volume $r_1 \cdots r_n$ and after applying T it has volume $\lambda_1 r_1 \cdots \lambda_n r_n = (\det T) r_1 \cdots r_n$.
- Then we use *analysis* to get the volume of Ω with a bunch of infinitesimally small boxes.

Determinant and Volume

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Prop'n:

Suppose $T \in \mathcal{L}(\mathbb{R}^n)$ is a any operator and $\Omega \subset \mathbb{R}^n$. Then

$$\text{volume } T(\Omega) = |\det T|(\text{volume } \Omega).$$

References

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