Lecture 27: Review

MATH 110-3

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August 8-9, 2023

Terms

- Eigenvalue
- Eigenvector
- Polynomials of operators
- Upper triangular matrix, diagonal matrix
- Inner product, properties
- Euclidean inner product
- Norm, basic properties
- Orthogonal
- Orthonormal
- Orthogonal complement, basic properties
- Orthogonal projection, basic properties

Terms

- Adjoint
- Self-adjoint operator
- Normal operator
- Positive operator
- Square root
- Isometries
- Nilpotent operator

Terms

- Generalized eigenvector, generalized eigenspace
- Algebraic multiplicity
- Geometric multiplicity
- Block diagonal matrix
- Characteristic polynomial
- Minimal polynomial
- Jordan basis
- Trace, basic properties
- Determinant, basic properties

Results/Tools

- Equivalent conditions to be an eigenvector (Axler 5.6)
- Linear independence of eigenvectors
- Every operator on a complex vector space has an eigenvalue
- Conditions for an upper triangular matrix
- Over $\mathbb{C}$, every matrix has an upper-triangular form with respect to some basis
- What does upper-triangular form tell us about invertibility, eigenvalues?
- Conditions for diagonalizability

Conditions for an upper triangular matrix

5.26 Conditions for upper-triangular matrix

Suppose $T \in \mathcal{L}(V)$ and $v_1, \ldots, v_n$ is a basis of V. Then the following are equivalent:

(a) the matrix of T with respect to $v_1, \ldots, v_n$ is upper triangular;

(b) $Tv_j \in \operatorname{span}(v_1, \ldots, v_j)$ for each $j = 1, \ldots, n$;

(c) $\operatorname{span}(v_1, \ldots, v_j)$ is invariant under T for each $j = 1, \ldots, n$.

Conditions for diagonalizability

5.41 Conditions equivalent to diagonalizability

Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Let $\lambda_1, \ldots, \lambda_m$ denote the distinct eigenvalues of T. Then the following are equivalent:

(a) T is diagonalizable;

(b) V has a basis consisting of eigenvectors of T;

(c) there exist 1-dimensional subspaces $U_1, \ldots, U_n$ of V, each invariant under T, such that

$$V = U_1 \oplus \cdots \oplus U_n;$$

(d) $V = E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_m, T)$;

(e) $\dim V = \dim E(\lambda_1, T) + \cdots + \dim E(\lambda_m, T)$.

Results/Tools

- Pythagorean theorem
- Orthogonal decomposition:
  Set $c = \dfrac{\langle u, v\rangle}{||v||^2}$ and $w = u - cv$. Then $\langle w, v\rangle = 0$ and $u = cv + w$.
- Cauchy-Schwarz
- Triangle-Inequality
- Norm of a linear combination (Axler 6.25)
- Writing a vector as a linear combination of an orthonormal basis (Axler 6.30)
- Gram-Schmidt
- Existence of orthonormal basis
- Schur's theorem
- Riesz Representation Theorem
- $V = U \oplus U^{\perp}$

Results/Tools

- Matrix of T^* with respect to an orthonormal basis
- Eigenvalues of self-adjoint operators are real
- Normal if and only if $||Tv|| = ||T^*v||$
- Complex and real spectral theorems
- Characterizing positive operators
- Characterizing isometries

Characterizing positive operators

7.35 Characterization of positive operators

Let $T \in \mathcal{L}(V)$. Then the following are equivalent:

(a) T is positive;

(b) T is self-adjoint and all the eigenvalues of T are nonnegative;

(c) T has a positive square root;

(d) T has a self-adjoint square root;

(e) there exists an operator $R \in \mathcal{L}(V)$ such that $T = R^* R$.

Characterizing isometries

7.42 Characterization of isometries

Suppose $S \in \mathcal{L}(V)$. Then the following are equivalent:

(a) S is an isometry;

(b) $\langle Su, Sv \rangle = \langle u, v \rangle$ for all $u, v \in V$;

(c) $Se_1, \ldots, Se_n$ is orthonormal for every orthonormal list of vectors $e_1, \ldots, e_n$ in V;

(d) there exists an orthonormal basis $e_1, \ldots, e_n$ of V such that $Se_1, \ldots, Se_n$ is orthonormal;

(e) $S^*S = I$;

(f) $SS^* = I$;

(g) S^* is an isometry;

(h) S is invertible and $S^{-1} = S^*$.

Results/Tools

- Increasing sequence of null spaces and termination (Axler 8.2-8.4)
- $V = \text{null } T^{\dim V} \oplus \text{range } T^{\dim V}$
- $G(\lambda, T) = \text{null } (T - \lambda I)^{\dim V}$
- Matrix of a nilpotent operator
- Description of operators on complex vector spaces
- Over $\mathbb{C}$, invertible operators have square roots
- Cayley-Hamilton
- Eigenvalues are zeros of minimal polynomial
- Jordan Form exists for any $T \in \mathcal{L}(V)$ where V is complex

Description of operators on complex vector spaces

8.21 Description of operators on complex vector spaces

Suppose V is a complex vector space and $T \in \mathcal{L}(V)$. Let $\lambda_1, \dots, \lambda_m$ be the distinct eigenvalues of T. Then

(a) $V = G(\lambda_1, T) \oplus \cdots \oplus G(\lambda_m, T)$;

(b) each $G(\lambda_j, T)$ is invariant under T;

(c) each $(T - \lambda_j I)|_{G(\lambda_j, T)}$ is nilpotent.

Practice Questions

1. Prove that the orthogonal projection map is self-adjoint.

Let $v, w \in V$. Write $v = u_1 + u_1'$ and $w = u_2 + u_2'$ such that $u_1, u_2 \in U$ and $u_1', u_2' \in U^\perp$. Then

$$\begin{aligned}
\langle P_U(v), w \rangle &= \langle u_1, w \rangle \\
&= \langle u_1, u_2 + u_2' \rangle \\
&= \langle u_1, u_2 \rangle + \langle u_1, u_2' \rangle \\
&= \langle u_1, u_2 \rangle \\
&= \langle u_1, u_2 \rangle + \langle u_1', u_2 \rangle \\
&= \langle u_1 + u_1', u_2 \rangle \\
&= \langle v, P_U(w) \rangle
\end{aligned}$$

Practice Questions

2. Fix a positive integer n. In the inner product space of continuous real-valued functions on $[-\pi, \pi]$ with inner product

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(x) g(x) dx,$$

let $V = \text{span}(1, \cos x, \cos 2x, \dots, \cos nx, \sin x, \sin 2x, \dots, \sin nx)$.

   2.1 Define $D \in \mathcal{L}(V)$ by $Df = f'$. Show $D^* = -D$. Conclude that D is normal but not self-adjoint.

Integration by parts and the fact that $f(\pi) = f(-\pi)$ for all vectors in the vector space gives that

$$\langle Df, g \rangle = -\int_{-\pi}^{\pi} g'(x) f(x) = -\langle f, Dg \rangle.$$

Practice Questions

3. Suppose T is the operator corresponding to the following matrix.

$$\begin{pmatrix} 3 & -1 & -2 \\ 0 & 2 & -3 \\ 0 & 0 & 2 \end{pmatrix}$$

3.1 Find the eigenvalues of T. 3,2

3.2 Find the characteristic and minimal polynomials of T.

char poly: $(z-3)(z-2)^2$ min poly: also $(z-3)(z-2)^2$

3.3 Find a basis of generalized eigenvectors.

from $G(3,T) = E(3,T)$ get $(1,0,0)$, from $G(2,T) \neq E(2,T)$ get $(-1,1,0),(1,0,1)$

3.4 Find the Jordan Form of T.

$$\begin{pmatrix} (3) & \\ & \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \end{pmatrix}$$

Practice Questions

4. Suppose T is the operator corresponding to the following matrix.

$$\begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & 5 \\ 0 & 0 & 3 \end{pmatrix}$$

   4.1 Find the eigenvalues of T. 1,3
   4.2 Find the characteristic and minimal polynomials of T. char poly: $(z-1)^2(z-3)$, min poly: $(z-1)(z-3)$
   4.3 Find a basis of generalized eigenvectors.

   from $G(1,T) = E(1,T)$ get $(1,0,0)$ and $(0,1,0)$ from $G(3,T) = E(3,T)$ get $(5/2, 5/2, 1)$
   4.4 Find the Jordan Form of T.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

Practice Questions

5. Give an example of a matrix $A \in \mathbb{C}^{7,7}$ such that the following all hold:
   - A is not surjective
   - $A^5(A+3I)^4(A-4I)^4 = 0$
   - The minimal and characteristic polynomials are equal.
   - The trace is -1.

$$\begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & & \\ & \begin{pmatrix} -3 & 1 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & -3 \end{pmatrix} & \\ & & \begin{pmatrix} -4 & 1 \\ 0 & -4 \end{pmatrix} \end{pmatrix}$$

Practice Questions

6. Let T be an operator on a finite-dimensional inner product space. Show that if $T^*T + TT^* = 0$, then $T = 0$.

$$
\begin{aligned}
0 &= \langle (T^*T + TT^*)v, v\rangle \\
&= \langle T^*Tv, v\rangle + \langle TT^*v, v\rangle \\
&= \langle Tv, Tv\rangle + \langle T^*v, T^*v\rangle \\
\implies Tv &= 0
\end{aligned}
$$

Practice Questions

7. Let T be an operator on a finite-dimensional inner product space.

   7.1 Suppose that $\langle Tv, v\rangle > 0$ for all nonzero $v \in V$. Show that every eigenvalue of T^2 is a positive real number.

   Let $\lambda \in \mathbb{F}$ such that $T^2 v = \lambda v$ for some $v \in V$. Then

$$\begin{aligned} 0 &< \langle T^2 v, Tv\rangle && (1)\\ &= \langle \lambda v, Tv\rangle && (2)\\ &= \lambda\langle v, Tv\rangle && (3)\\ &= \lambda\langle Tv, v\rangle && (4) \end{aligned}$$

   which implies λ must be positive and real.

Practice Questions

8. Prove that the linear operator on $\mathbb{C}^3$ defined by the matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$ is an isometry.

This operator is the one that sends $(x, y, z) \in \mathbb{C}^3$ to $(x, -z, y) \in \mathbb{C}^3$. The Eucliedean inner product gives

$$||(x, y, z)||^2 = x \cdot \overline{x} + y \cdot \overline{y} + z \cdot \overline{z} = x \cdot \overline{x} + -z \cdot \overline{-z} + y \cdot \overline{y} = ||(x, -z, y)||^2.$$

Practice Questions

9. What are some things that are special about orthonormal bases?

$$v = \langle v, e_1 \rangle e_1 + \ldots \langle v, e_n \rangle e_n$$

for any $v \in V$ and

$$||v||^2 = |\langle v, e_1 \rangle|^2 + \ldots |\langle v, e_n \rangle|^2$$

References

[Axl14] Sheldon Axler.
Linear Algebra Done Right.
Undergraduate Texts in Mathematics. Springer Cham, 2014.