Lecture 2: Subspaces

MATH 110-3

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- You can earn a 90% for H,D,Q but never taking a quiz or showing up for a discussion.

Last Time...

Last class:

- Fields $\mathbb{F}$, such as $\mathbb{R}, \mathbb{C}$
- $\mathbb{F}$-Vector Spaces, such as $\mathbb{F}^n, \mathbb{F}^\infty, \mathbb{F}^S$

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1.19 **Definition** ***vector space***

A ***vector space*** is a set V along with an addition on V and a scalar multiplication on V such that the following properties hold:

commutativity
$u + v = v + u$ for all $u, v \in V$;

associativity
$(u + v) + w = u + (v + w)$ and $(ab)v = a(bv)$ for all $u, v, w \in V$ and all $a, b \in \mathbf{F}$;

additive identity
there exists an element $0 \in V$ such that $v + 0 = v$ for all $v \in V$;

additive inverse
for every $v \in V$, there exists $w \in V$ such that $v + w = 0$;

multiplicative identity
$1v = v$ for all $v \in V$;

distributive properties
$a(u + v) = au + av$ and $(a + b)v = av + bv$ for all $a, b \in \mathbf{F}$ and all $u, v \in V$.

Subspace Criteria

Prop'n:

$U \subseteq V$ is a subspace **if and only if** U satisfies each of the following:

1. **additive identity:** $0 \in U$
2. **closed under addition:** $u, w \in U$ implies $u + w \in U$
3. **closed under scaling:** $u \in U$, $\lambda \in \mathbb{F}$ implies $\lambda u \in U$

Proof

Lemmas:

Let V be a vector space and $v \in V$.

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Proof of b.

$$v + (-1)v = (1-1)v = 0v = 0$$

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S'pose $U \subseteq V$ is a subspace...

S'pose $U \subseteq V$ satisfies the three conditions...

Examples

1. $y = x^2$ in $\mathbb{R}^2$
2. $y = 3x$ in $\mathbb{R}^2$
3. $y = 3x \bigcup y = -2x$ in $\mathbb{R}^2$

Operations on Subspaces

Def'n:

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$U_1 = \{(x, 0, 0) : x \in \mathbb{R}\}$, $U_2 = \{(0, y, 0) : y \in \mathbb{R}\}$, and
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Prop'n:

S'pose $U_1, \ldots, U_n$ are subsets of V. $U_1 + \cdots + U_n$ is the smallest subspace of V containing each of the U_i.

Proof

To do:

1. Check that $U_1 + \cdots + U_n$ is a subspace.
2. Show that no smaller subspace contains each U_i.

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For (1), check subspace criteria.

For (2), show $U_i \in U_1 + \cdots + U_n$ by setting all but u_i equal to zero. Then because subspaces must contain all finite sums of their elements any subspace containing each of the U_i's contains $U_1 + \cdots + U_n$.

Direct Sums

Def'n:

The vector subspace $U_1 + \cdots + U_n$ is called a **direct sum** and denoted $U_1 \oplus \cdots \oplus U_n$ when each element of $U_1 + \cdots + U_n$ can be written in a unique way as $u_1 + \cdots + u_n$ for $u_i \in U_i$.

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Example: $U_1 = \{(x, 0, 0) : x \in \mathbb{R}\}$, $U_2 = \{(0, y, 0) : y \in \mathbb{R}\}$, and
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As a set, $U_1 + U_2 + U_3 = U_1 \oplus U_2 \oplus U_3 = \mathbb{R}^3$.

Non-example: $U_1 = \{(x, y, 0) : x, y \in \mathbb{R}\}$, $U_2 = \{(0, w, z) : w, z \in \mathbb{R}\}$
$\mathbb{R}^3 = U_1 + U_2 \neq U_1 \oplus U_2$.

Conditions for Direct Sum

Prop'n:

S'pose $U_1, \ldots, U_n$ are subspaces of V. Then $U_1 + \cdots + U_n$ is a direct sum if and only if the only way to write 0 as a sum $u_1 + \cdots + u_n$ is where each $u_i = 0 \in U_i$.

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Direct Sum of Two Subspaces

S'pose U and W are subspaces of V. Then $U + W$ is a direct sum if and only if $U \cap W = \{0\}$.

Example

S'pose $U = \{(x, x, y, y) \in \mathbb{F}^4 : x, y \in \mathbb{F}\}$. Find a subspace of $\mathbb{F}^4$ such that $\mathbb{F}^4 = U \oplus W$.

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Claim: $W = \{(z, 0, w, 0) : z, w \in \mathbb{F}^4\}$ works.

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Claim: $W = \{(z, 0, w, 0) : z, w \in \mathbb{F}^4\}$ works.

Break

Discussion Questions

1. Suppose $v, w \in V$. Prove there exists a unique $x \in V$ such that $v + 3x = w$. (Axler 1.B.3)
2. Define addition and multiplication on $\mathbb{R} \cup \{\infty\} \cup \{-\infty\}$ as:

$$t\infty = \begin{cases} -\infty & t < 0 \\ 0 & t = 0 \\ \infty & t > 0 \end{cases} \qquad t \cdot -\infty = \begin{cases} \infty & t < 0 \\ 0 & t = 0 \\ -\infty & t > 0 \end{cases}$$

$$t + \infty = \infty + t = \infty \quad t + (-\infty) = (-\infty) + t = -\infty$$

$$\infty + \infty = \infty \quad -\infty + -\infty = -\infty \quad \infty + -\infty = 0$$

Show which $\mathbb{R}$-vector space properties are satisfies or not.

Discussion Questions (Cont'd)

3. Prove the following:
   (a) The set of continuous real-valued functions on the interval $[0, 1]$ is a subspace of $\mathbb{R}^{[0,1]}$.
   (b) If $b \in \mathbb{F}$, then $\{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_3 = 5x_4 + b\}$ is a subspace of $\mathbb{R}^4$ if and only if $b = 0$.
4. Suppose $U = \{(x, y, x + y, x - y, 2x) \in \mathbb{F}^5 : x, y \in \mathbb{F}\}$. Find a subspace of $\mathbb{F}^5$ such that $\mathbb{F}^5 = U \oplus W$. (Axler 1.C.21)

Discussion Solutions

1. Using additive inverse of v and scaling by $\frac{1}{3}$, we have $x = \frac{1}{3}(w - v)$ which shows this element to be unique.
2. A property which fails is associativity, for example,

$$(t + \infty) + -\infty = \infty + -\infty = 0$$

$$t + (\infty + -\infty) = t + 0 = t$$

3. (a) The zero function is continuous. The sum of continuous functions is continuous. Scaling a continuous function by a real number is still continuous.
   (b) Subspace criteria hold if $b = 0$, if $b \neq 0$ both additive closure and closed under scalar multiplication fail.
4. $W = \{(0, 0, a, b, c) : a, b, c \in \mathbb{F}\}$ works. We verify this by seeing the $U \cap W = \{0\}$.

References

[Axl14] Sheldon Axler.
Linear Algebra Done Right.
Undergraduate Texts in Mathematics. Springer Cham, 2014.