Lecture 3: Span and Linear Independence

MATH 110-3

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Announcements

- Quiz today! 15 minutes after lecture
- Homework 2 - problem 5 W should be a V

Notation

- $\mathbb{F}$ will be $\mathbb{R}$ or
- V will be an $\mathbb{F}$-vector space

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- V will be an $\mathbb{F}$-vector space
- A list of vectors will be written $v_1, v_2, \ldots, v_n$
- By ***list*** of vectors we mean a finite collection of vectors

Linear Combinations

Def'n:

A **linear combination** of the vectors $v_1, \ldots, v_n$ is a vector

$$a_1 v_1 + \cdots + a_n v_n$$

for $a_i \in \mathbb{F}$.

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Example:

- $(3, 4, 5) = (3, 0, 2) + 2(0, 2, 1) + \frac{1}{2}(0, 0, 2)$
- Is $(3, 4, 5)$ a linear combination of $(3, 0, 2), (0, 2, 1)$?

Span

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The set of all linear combinations of the vectors $v_1, \ldots, v_n$ is called the span of $v_1, \ldots, v_n$.
The empty set has span $\{0\}$.

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Fact ([Axl14] 2.7):

The span of a list of vectors in V is the smallest subspace of V that contains all the vectors in the list.

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Claim:

The vectors $(1, 0, \ldots, 0)$, $(0, 1, \ldots, 0)$, $\ldots (0, 0, \ldots, 1)$ span $\mathbb{F}^n$.

How would I prove this?

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The vectors $(1, 0, \ldots, 0)$, $(0, 1, \ldots, 0)$, $\ldots (0, 0, \ldots, 1)$ span $\mathbb{F}^n$.

How would I prove this?

- Show vector $(x_1, \ldots, x_n) \in \mathbb{F}^n$ can be written as a linear combination of these vectors.

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$\mathcal{P}(\mathbb{R})$ is an **infinite-dimensional** vector space.

Def'n:

A vector space V is **infinite-dimensional** if it is not finite-dimensional.

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We wrote

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Is this the only set of coefficients that would work to write $(3, 4, 5)$ as a linear combination of this list?

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Then subtracting

$$0 = (a_1 - b_1)v_1 + \ldots + (a_n - b_n)v_n.$$

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$$0 = (a_1 - b_1)v_1 + \ldots + (a_n - b_n)v_n.$$

If the only way to write 0 as a linear combination of the v_i is where all the coefficients are 0, then we have uniqueness of *every* set of coefficients.

Linear Independence

Def'n

A list of vectors $v_1, \ldots, v_n$ is said to be **linearly independent** if the only choice of $a_i \in \mathbb{F}$ that makes $a_1 v_1 + \cdots + a_n v_n = 0$ is $a_1 = \cdots = a_n = 0$.

Otherwise, we say the vectors are **linearly dependent**.

The empty set is linearly independent.

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- A single vector v?
- Two vectors?
- $1, x, x^2, x^3, \ldots$ in $\mathcal{P}(\mathbb{R})$? *Independent.*
- A list of vectors containing the zero vector? *Always dependent.*

Linear Dependence Lemma

Prop'n 2.21 [Axl14]:

S'pose $v_1, \ldots, v_n$ is a linearly dependent list in V. Then there exists $j \in [n]$ such that the following hold:

- $v_j \in \text{span}(v_1, \ldots, v_{j-1})$
- $\text{span}(v_1, \ldots, v_{j-1}, v_{j+1}, \ldots, v_n) = \text{span}(v_1, \ldots, v_n)$

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In finite dimensional vector spaces, the length of a linear independent list is $\leq$ the length of a spanning list.

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Proof. S'pose $v_1, \ldots, v_m$ are lin ind and $u_1, \ldots, u_n$ span. WTS: $m \leq n$.

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By (2.21), we can remove some u_i such that

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And we repeat...

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$$\text{n-j of the } u's, v_1, v_2, \ldots, v_j$$

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We know that we can choose one of the $u's$ to replace at each step because the $v's$ are lin ind.

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$$\text{\# of v's} \leq \text{\# of u's}$$

$$m \leq n$$

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Example:

We know $(1,2,3),(2,3,4),(3,4,5),(1,1,1)$ is linearly dependent in $\mathbb{R}^3$ because $(1,0,0),(0,1,0),(0,0,1)$ spans $\mathbb{R}^3$.

Finite-dimensional subspaces

Prop'n 2.26 [Axl14]:

Every subspace of a finite-dimensional vector space is finite-dimensional.

References

[Axl14] Sheldon Axler.
Linear Algebra Done Right.
Undergraduate Texts in Mathematics. Springer Cham, 2014.