Lecture 4: Bases

MATH 110-3

Franny Dean

June 26, 2023

Announcements

MATH 110: Linear Algebra

Summer 2023

Instructor:	Franny Dean (she/her/hers)	**Times:**	MW 4:10 – 6 pm T 4:10 – 5 pm Th 4:10 – 5:20 pm
Email:	frances_dean@berkeley.edu	**Place:**	Cory 241

Course Page: `http://frances-dean.github.io` ⇒ Teaching ⇒ MATH 110

Office Hours: Tuesday 1:30-2:30pm Evans 748; Tuesday 5:00-6:00 pm Cory 241; Wednesday 12:00-1:00pm Evans 732

Last Time...

- Linearly independent lists
- Spanning lists

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- length spanning list $\geq$ length of linearly independent list

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$$x = y - z$$
$$y = y$$
$$z = z$$

Criterion for Bases

Prop'n 2.29 [Axl14]:

A list $v_1, \ldots, v_n$ of vectors in V is a basis if and only if for every $v \in V$ there are unique scalars $a_i \in \mathbb{F}$ such that $v = a_1 v_1 + \ldots + a_n v_n$.

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Hence, B is a basis. $\square$

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Example

Let $U \subset \mathbb{C}^5$ defined as

$$U = \{(z_1, z_2, z_3, z_4, z_5) \in \mathbb{C}^5 : 6z_1 = z_2 \text{ and } z_3 + 2z_4 + 3z_5 = 0\}.$$

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So:

$$(z_1, z_2, z_3, z_4, z_5) = z_1(1, 6, 0, 0, 0) + z_4(0, 0, -2, 1, 0) + z_5(0, 0, -3, 0, 1).$$

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Add standard basis vectors!

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$$(a, 0, b, 0, 0) \neq (c_1, 6c_2, -2c_2 - 3c_3, c_2, c_3)$$

Subspaces are part of direct sums to V

Prop'n 2.34 [Axl14]

Suppose V is finite-dimensional and U is a subspace of V. Then there is a subspace W of V such that $V = U \oplus W$.

Break

References

[Axl14] Sheldon Axler.
Linear Algebra Done Right.
Undergraduate Texts in Mathematics. Springer Cham, 2014.