Lecture 5: Dimension

MATH 110-3

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But, we have ∞-ly many bases...

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We **need** first:

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On the other hand, since B' spans and B is linearly independent:

$$\text{length}(B') \geq \text{length}(B).$$

$\square$

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If V is finite-dimensional and U is a subspace, then $\dim U \leq \dim V$.

Proof. Let $U \subseteq V$ as in the proposition.
Choose B a basis for V and C a basis for U.
Then B spans V and C is linearly independent in U, thus

$$\text{length}(B) \geq \text{length}(C).$$

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$\dim U = 0$: $\{0\}$
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The only subspace with the same dimension as V is V.

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But since there are $\dim V$ vectors, and every basis has length n the extension is trivial. $\square$

Corollary

Every list of linearly independent vectors in finite-dimensional V with length $\dim V$ is a basis.

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How do we know that $\dim U < \dim \mathcal{P}_4(\mathbb{R})$?

Spanning Lists of Length n

Prop'n:

Let $\dim V = n$. Then if $v_1, \ldots, v_n$ is spans, it forms a basis.

Dimension of a Sum

Prop'n (2.43):

If U_1 and U_2 are subspaces of a finite-dimensional vector space,

$$\dim(U_1 + U_2) = \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2).$$

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Why does this make sense?

What does this mean for direct sums?

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$$\begin{aligned}
&\implies \dim U \cap W = 0 \\
&\implies U \cap W = \{0\} \\
&\implies U + W = U \oplus W = \mathbb{R}^8
\end{aligned}$$

References

[Axl14] Sheldon Axler.
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Undergraduate Texts in Mathematics. Springer Cham, 2014.