Lecture 6: Linear Maps

MATH 110-3

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Context

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> "No one gets excited about vector spaces. The interesting part of linear algebra is the subject to which we now turn−linear maps."
> - *S. Axler*

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Def'n:

The set of all linear maps V to W is denoted $\mathcal{L}(V, W)$.

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- From $\mathbb{R}^3 \to \mathbb{R}^2$: $T(x, y, z) = (2x - y, 5x - 7z, 2z - y)$.
- Generalizing from $\mathbb{F}^n \to \mathbb{F}^m$:
  Let $A_{j,k} \in \mathbb{F}$ for $j \in [m]$, $k \in [n]$, define $T \in \mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$ as

$$T(x_1, \ldots, x_n) = (A_{1,1}x_1 + \ldots + A_{1,n}x_n, \ldots, A_{m,1}x_1 + \ldots + A_{m,n}x_n).$$

Basis of Domain

Prop'n 3.5:

Suppose $v_1, \ldots, v_n$ is a basis of V and $w_1, \ldots, w_n \in W$. Then there exists a unique linear map $T : V \to W$ such that

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2. A linear map is uniquely determined by where it sends basis vectors.

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Define $T : V \to W$ by

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for arbitrary $c_1, \ldots, c_n \in \mathbb{F}$.

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$$\begin{aligned} T(x+y) &= T((a_1 + b_1)v_1 + \ldots (a_n + b_n)v_n) \\ &= (a_1 + b_1)w_1 + \ldots (a_n + b_n)w_n \\ &= (a_1 w_1 + \ldots a_n w_n) + (b_1 w_1 + \ldots b_n w_n) \\ &= Tx + Ty \end{aligned}$$

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Done with **existence!**

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Then additivity implies $T(c_1 v_1 + \ldots + c_n v_n) = c_1 w_1 + \ldots + c_n w_n$.

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Then additivity implies $T(c_1 v_1 + \ldots + c_n v_n) = c_1 w_1 + \ldots + c_n w_n$.

Thus, T must be the map we just defined.

Algebra of $\mathcal{L}(V, W)$

Def'n:

Let $S, T \in \mathcal{L}(V, W)$ and $\lambda \in \mathbb{F}$. We define for all $v \in V$

$$(S + T)(v) = Sv + Tv$$

and

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Prop'n:

With the addition and scalar multiplication defined above, $\mathcal{L}(V, W)$ is a **vector space**.

More algebra of $\mathcal{L}(V, W)$

Def'n:

Let U, V, W be vector spaces. If $T \in \mathcal{L}(U, V)$, $S \in \mathcal{L}(V, W)$, then we define $ST \in \mathcal{L}(U, W)$ for $u \in U$ as

$$(ST)(u) = S(Tu).$$

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⚠ Notice that $TD \neq DT$.

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for $T \in \mathcal{L}(V, W)$

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- **identity:**

$$T I_V = I_W T = T$$

  for $T \in \mathcal{L}(V, W)$

- **distributive properties:**

$$(S_1 + S_2)T = S_1 T + S_2 T$$

$$S(T_1 + T_2) = ST_1 + ST_2$$

  where $T's \in \mathcal{L}(U, V)$ and $S's \in \mathcal{L}(V, W)$

Break

Discussion Questions

1. Prove that all linear maps send 0 to 0.
2. Describe the subspaces of $\mathbb{R}^3$ and their dimensions.
3. Let $U = \{p \in \mathcal{P}_4(\mathbb{R}) : p''(6) = 0\}$.
   (a) Find a basis of U.
   (b) Extend your basis to one of $\mathcal{P}_4(\mathbb{R})$.
   (c) Find a subspace W such that $\mathcal{P}_4(\mathbb{F}) = U \oplus W$.
4. Suppose $b, c \in \mathbb{R}$. Define $T : \mathbb{R}^3 \to \mathbb{R}^2$ by $T(x, y, z) = (2x - 4y + 3z + b, 6x + cxyz)$. Show that T is linear if and only if $b = c = 0$.
5. Construct a linear map $T : \mathbb{R}^3 \to \mathbb{R}^2$ such that $T(3, 0, 2) = (3, 4)$ and $T(1, 1, 1) = (1, 1)$. Or show why this is impossible. what about one such that $T(3, 0, 2) = (3, 4)$ and $T(6, 0, 4) = (1, 1)$?

Discussion Question Hints/Solutions

1. $T(0) = T(0+0) = T(0) + T(0)$. Adding the additive inverse of $T(0)$ to each side gives $0 = T(0)$.

2.

dim	$U \subseteq \mathbb{R}^3$
0	$\{0\}$
1	all lines in $\mathbb{R}^3$ through the origin
2	all planes in $\mathbb{R}^3$ through the origin
3	$\mathbb{R}^3$

3. (a) Show that the following are linearly ind. and span: $1, (x-6), (x-6)^3, (x-6)^4$
   (b) The polynomial $(x-6)^2$ is not in U and this list is lin ind of the right length.
   (c) Let $W = \text{span}((x-6)^2)$.

Discussion Question Hints/Solutions

4. First, if $b = c = 0$. We can show that T is additive and satisfies homogeneity.
   Next, notice that
   $\lambda T(x, y, z) \neq T(\lambda x, \lambda y, \lambda z) = (2\lambda x - 4\lambda y + 3\lambda z + b, 6\lambda x + c\lambda^3 xyz)$
   for any $\lambda \in \mathbb{R}$.
5. We can use the basis of domain prop'n to define a map in the first case. One example is to say $T(3, 0, 2) = (3, 4)$, $T(1, 1, 1) = (1, 1)$, and $T(0, 1, 0) = (2, 1)$ (we needed to choose a third vector and output to have a basis of the domain $\mathbb{R}^3$). The second will not be linear because $(3, 0, 2) = \frac{1}{2}(6, 0, 4)$ but $T(3, 0, 2) \neq \frac{1}{2}T(6, 0, 4)$.

References

[Axl14] Sheldon Axler.
Linear Algebra Done Right.
Undergraduate Texts in Mathematics. Springer Cham, 2014.