Lecture 7: Rank-Nullity

MATH 110-3

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June 29, 2023

General Functions

- Well-defined
- Injective
- Surjective

Null Space

Def'n:

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⚠ the null space is also called the **kernel** ⚠

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- For $u \in$ null T, $\lambda \in \mathbb{F}$. Then

$$T(\lambda u) = \lambda Tu = \lambda 0 = 0$$

  and $\lambda u \in$ null T.

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Tells us that only multiplication by x^2 is injective.

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- $\{0\} \subseteq$ null T because null T is a subspace
- null $T \subseteq \{0\}$ because for $v \in$ null V,

$$T(v) = 0 = T(0)$$

  injectivity implies $v = 0$
- $\implies \{0\} =$ null T

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- $\implies u - v \in \text{null } T = \{0\}$ and $u = v$ $\square$

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⚠ the range is also called the **image** ⚠

Range is a Subspace of W...

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Fundamental Theorem of Linear Maps

Prop'n:

Let V be a finite-dimensional vector space and $T \in \mathcal{L}(V, W)$. Then range T is finite dimensional and

$$\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T.$$

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⚠ This result is *usually* called **rank-nullity**! ⚠

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Done.

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$$\begin{aligned} \dim \operatorname{null} T &= \dim V - \dim \operatorname{range} T \\ &\geq \dim V - \dim W \\ &> 0 \end{aligned}$$

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$$< \dim W$$

Consequences for systems of linear equations

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Homogeneous System of Linear Equations:

$$\sum_{k=1}^{n} A_{1,k} x_k = 0$$

$$\cdots$$

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Is the null space strictly bigger than $\{0\}$*?*

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Question becomes is the range all of $\mathbb{F}^m$?

References

[Axl14] Sheldon Axler.
Linear Algebra Done Right.
Undergraduate Texts in Mathematics. Springer Cham, 2014.