Lecture 8: Matrices

MATH 110-3

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July 3, 2023

Motivation

Last lecture...

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Systems of Linear Equations:

$$\sum_{k=1}^{n} A_{1,k} x_k = c_1$$

$$\cdots$$

$$\sum_{k=1}^{n} A_{m,k} x_k = c_m$$

$A_{i,j} \in \mathbb{F}$

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Turn into a linear map question:

$$T : \mathbb{F}^n \to \mathbb{F}^m \qquad T(x_1, \dots, x_n) = (\sum_{k=1}^{n} A_{1,k} x_k, \dots, \sum_{k=1}^{n} A_{1,k} x_k)$$

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$$\begin{aligned} 5x_1 + 2x_2 + 3x_3 &= c_1 \\ x_1 + 6x_2 + x_3 &= c_2 \\ 2x_1 + 2x_2 + x_3 &= c_3 \end{aligned}$$

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Becomes

$$\mathcal{M}(T) = \begin{pmatrix} 5 & 2 & 3 \\ 1 & 6 & 1 \\ 2 & 2 & 1 \end{pmatrix}$$

Matrices

Def'n:

A $m \times n$ **matrix** is a rectangular array of elements of $\mathbb{F}$ with m rows and n columns:

$$\begin{pmatrix} A_{1,1} & \dots & A_{1,n} \\ \vdots & & \vdots \\ A_{m,1} & \dots & A_{m,n} \end{pmatrix}$$

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- Rows by Columns

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Reminders

- Rows by Columns
- Entry in i-row, j-column is $A_{i,j}$

Matrix of a Linear Map

Def'n:

S'pose $T \in \mathcal{L}(V, W)$ and $v_1, \ldots, v_n$ is a basis of V and $w_1, \ldots, w_n$ a basis of W. The **matrix of** T with respect to these bases is $\mathcal{M}(T)$ where $A_{j,k}$ defined by

$$Tv_k = A_{1,k}w_1 + \ldots + A_{m,k}w_m.$$

Matrix of Linear Map: $\mathbb{F}^n \to \mathbb{F}^m$

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Write the matrix of $T \in \mathcal{L}(\mathbb{R}^3, \mathbb{R}^3)$ for

$$T(x_1, x_2, x_3) = (5x_1 + 2x_2 + 3x_3, x_1 + 6x_2 + x_3, 2x_1 + 2x_2 + x_3)$$

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Notice that $T(1,0,0) = (5,1,2)$, $T(0,1,0) = (2,6,2)$, and $T(0,0,1) = (3,1,1)$.

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$$\mathcal{M}(T) = \begin{pmatrix} 5 & 2 & 3 \\ 1 & 6 & 1 \\ 2 & 2 & 1 \end{pmatrix}$$

Matrix of Differentiation

$D \in \mathcal{L}(\mathcal{P}_3(\mathbb{R}), \mathcal{P}_2(\mathbb{R}))$ is the differentiation map defined $Dp = p'$

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We use the bases: $1, x, x^2, x^3$ of $\mathcal{P}_3(\mathbb{R})$ and $1, x, x^2$ of $\mathcal{P}_2(\mathbb{R})$:

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$$
\begin{aligned}
D(1) &= 0 && = 0 \cdot 1 + 0 \cdot x + 0 \cdot x^2 \\
D(x) &= 1 && = 1 \cdot 1 + 0 \cdot x + 0 \cdot x^2 \\
D(x^2) &= 2x && = 0 \cdot 1 + 2 \cdot x + 0 \cdot x^2 \\
D(x^3) &= 3x^2 && = 0 \cdot 1 + 0 \cdot x + 3 \cdot x^2
\end{aligned}
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D(x^3) &= 3x^2 && = 0 \cdot 1 + 0 \cdot x + 3 \cdot x^2
\end{aligned}$$

$$\mathcal{M}(D) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

Matrix of Integration

$T \in \mathcal{L}(\mathcal{P}_3(\mathbb{R}), \mathbb{R})$ is the integration map defined $Tp = \int_0^1 p(x)dx$

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We use the bases: $1, x, x^2, x^3$ of $\mathcal{P}_3(\mathbb{R})$ and 1 of $\mathbb{R}$:

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\begin{aligned}
T(1) &= \int_0^1 1dx && = 1 = 1 \cdot 1 \\
T(x) &= \int_0^1 xdx && = 1/2 = 1/2 \cdot 1 \\
T(x^2) &= \int_0^1 x^2dx && = 1/3 = 1/3 \cdot 1 \\
T(x^3) &= \int_0^1 x^3dx && = 1/4 = 1/4 \cdot 1
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$$\mathcal{M}(D) = \begin{pmatrix} 1 & 1/2 & 1/3 & 1/4 \end{pmatrix}$$

Matrix Algebra

Def'n:

The **sum** of two matrices is computed by adding corresponding entries.

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Prop'n 3.36:

Let $S, T \in \mathcal{L}(V, W)$. Then $\mathcal{M}(S + T) = \mathcal{M}(S) + \mathcal{M}(T)$.

Matrix Algebra (Cont'd)

Def'n:

The **scalar multiplication** of a matrix is computed by scaling each entry.

$$(\lambda A)_{i,j} = \lambda A_{i,j}$$

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Prop'n 3.36:

Let $\lambda \in \mathbb{F}$, $T \in \mathcal{L}(V, W)$. Then $\mathcal{M}(\lambda T) = \lambda \mathcal{M}(T)$.

Vector Space of Matrices

Def'n:

The set of all $m \times n$ matrices with entries in $\mathbb{F}$ are denoted $\mathbb{F}^{m,n}$.

Prop'n 3.40 [Axl14]:

For positive integers m, n, with matrix addition and scalar multiplication $\mathbb{F}^{m,n}$ is a vector space with dimension mn.

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What is the basis?

Review of Matrix Multiplication

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First, for $A \cdot B$, need A to be $m \times n$ and B to be $n \times l$ for any positive integrs m, n, l.

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$$\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix}$$

Multiplication of Matrices Corresponds to Composition of Linear Maps

$T : U \to V$, $S : V \to W$, then $ST : U \to W$

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We want $\mathcal{M}(ST) = \mathcal{M}(S)\mathcal{M}(T)$.

Multiplication of Matrices Corresponds to Composition of Linear Maps

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We want $\mathcal{M}(ST) = \mathcal{M}(S)\mathcal{M}(T)$.

We will ***choose*** to define matrix multiplication so that this is true.

Definition Matrix Multiplication

Def'n:

Suppose A is an $m \times n$ matrix and C is an $n \times p$ matrix. Then AC is defined to be the $m \times p$ matrix whose entry in row j column k is given by

$$(AC)_{j,k} = \sum_{r=1}^{n} A_{j,r} C_{r,k}.$$

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The entry in row j, column k of AC is computed by taking row j of A and column k of C multiplying corresponding entries and summing.

Product of Linear Maps

Prop'n 3.43 [Axl14]:

If $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$, then $\mathcal{M}(ST) = \mathcal{M}(S)\mathcal{M}(T)$.

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Proof sketch.

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Proof sketch. Let $\mathcal{M}(S) = A$, $\mathcal{M}(T) = C$.

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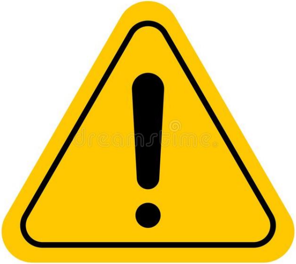
Proof sketch. Let $\mathcal{M}(S) = A$, $\mathcal{M}(T) = C$.

$$
\begin{aligned}
(ST)u_k &= S(\sum_{r=1}^{n} C_{r,k} v_r) && \text{use def'n matrix of map} \\
&= \sum_{r=1}^{n} C_{r,k} S v_r && \text{using linearity} \\
&= \sum_{r=1}^{n} C_{r,k} \sum_{j=1}^{m} A_{j,r} v_r && \text{use def'n matrix of map} \\
&= \sum_{r=j}^{m} (\sum_{r=1}^{n} A_{j,r} C_{r,k}) w_j && \text{rearranging the sums}
\end{aligned}
$$

Matrix Multiplication Does Not Commute



Matrix Multiplication Does Not Commute!

Other ways of thinking about matrix product

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Notation: A an $m \times n$ matrix

- $A_{j,\cdot}$ is the row j of A (a $1 \times n$ matrix)
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Other ways of thinking about matrix product (Cont'd)

Example.

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Prop'n:

Let $c = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$. Then

$$Ac = c_1 A_{\cdot,1} + \ldots + c_n A_{\cdot,n}.$$

References

[Axl14] Sheldon Axler.
Linear Algebra Done Right.
Undergraduate Texts in Mathematics. Springer Cham, 2014.