Lecture 9: Isomorphisms

MATH 110-3

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Invertible

Def'n:

A linear map $T \in \mathcal{L}(V, W)$ is called **invertible** if there exists a map $S \in \mathcal{L}(W, V)$ such that ST is the identity on V and TS is the identity on W.

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Notation

Write the inverse of T as T^{-1}.

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To complete the proof... show T^{-1} is linear.

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- Multiplication by x^2 from $\mathcal{P}(\mathbb{R})$ to itself.
- Backwards shift map $\mathbb{F}^\infty \to \mathbb{F}^\infty$.
- $T(x, y) = (0, y)$

Isomorphisms

Def'n:

An invertible linear map is also called an **isomorphism**.

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Two vector spaces are called **isomorphic** is there exists an isomorphism between them.

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T is injective, so null $T = 0$. T is surjective, so range $T = W$. Thus, by the formula

$$\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T,$$

we have

$$\dim V = 0 + \dim W.$$

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The map

$$T(c_1 v_1 + \ldots + c_n v_n) = c_1 w_1 + \ldots + c_n w_n$$

is an isomorphism. $\square$

Maps and Matrices

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Suppose $v_1, \ldots, v_m$ is a basis of V and $w_1, \ldots, w_n$ is a basis of W. Then $\mathcal{M}$ is an isomorphism between $\mathcal{L}(V, W)$ and $\mathbb{F}^{n,m}$.

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- $\mathcal{M}$ is linear
- $\mathcal{M}$ is injective
- $\mathcal{M}$ is surjective

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Remarkable result...

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Suppose V is *finite-dimensional* and $T \in \mathcal{L}(V)$. Then the following are equivalent:

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Break

Discussion Questions

1. Suppose $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$ are both invertible linear maps. Prove that $ST \in \mathcal{L}(U, W)$ is invertible and that its inverse is $(ST)^{-1} = T^{-1}S^{-1}$.
2. Come up with an example of a linear map that is injective but not surjective and one that is surjective but not injective.
3. Suppose $D \in \mathcal{L}(\mathcal{P}_3(\mathbb{R}), \mathcal{P}_2(\mathbb{R}))$ is the differentiation map defined $Dp = p'$. Find a basis of $\mathcal{P}_3(\mathbb{R})$ and a basis of $\mathcal{P}_2(\mathbb{R})$ such that the matrix of D with respect to these bases is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Discussion Questions

4. Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove there exists a basis of V and a basis of W such that with respect to these bases, all entries of $\mathcal{M}(T)$ are 0 except the entries in row j, column j equal 1 for all j such that $1 \leq j \leq \dim \operatorname{range} T$.
5. Find two 2×2 matrices that do not commute.
6. Let $a_0, \ldots, a_n$ be any scalars in $\mathbb{F}$. Consider the basis $1, x, \ldots, x^n$ of $\mathcal{P}(\mathbb{F})$ and the standard basis of $\mathbb{F}^{n+1}$. Write the matrix of the transformation

$$T : \mathcal{P}_n(\mathbb{F}) \to \mathbb{F}^{n+1}$$

$$p \to (p(a_0), p(a_1), \ldots, p(a_n))$$

with respect to these bases.

Discussion Questions

7. Suppose V is finite dimensional and $S, T, U \in \mathcal{L}(V)$ and $STU = I$. Show that T is invertible and that $T^{-1} = US$.
8. Suppose V is finite dimensional and $R, S, T \in \mathcal{L}(V)$ are such that RST is surjective. Prove S is injective.

Discussion Question Hints/Solutions

1.
$$(ST)(T^1S^{-1}) = S(TT^{-1})S^{-1} = SIS^{-1} = SS^{-1} = I$$
$$(T^{-1}S^{-1})(ST) = T^{-1}(S^{-1}S)T = T^{-1}IT = T^{-1}T = I$$

2. One solution is example 3.68 in [Axl14].

3. Basis for $\mathcal{P}_3(\mathbb{R})$: $x^3, x^2, x, 1$
   Basis for $\mathcal{P}_2(\mathbb{R})$: $3x^2, 2x, 1$

5. An example of two matrices that do not commute are

$$\begin{pmatrix} 2 & 4 \\ 3 & 5 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$$

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\end{aligned}$$

...giving the desired matrix form.

Discussion Question Hints/Solutions

6. This is the Vandermonde matrix.
7. Using prop'n from lecture, $(ST)U = I$ implies $U(ST) = I$. This implies $US = T^{-1}$.
8. RST surjective implies $R(ST)$ invertible. $R(ST)$ invertible implies ST invertible. ST invertible implies S is invertible which implies S is surjective.

References

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