MATH 110: Linear Algebra

Homework 4

Instructor Franny Dean

Instructions: Please type your solutions to the following in LaTeX and upload your solutions to Gradescope by 4:10pm on **Wednesday, July 12, 2023**. You are highly encouraged to work with your classmates, but your write up must be done independently without looking at any other student's solutions. One of these six problems will be randomly chosen to not be graded.

1. (Axler 3.C.14) Prove that matrix multiplication is associative. In other words, suppose A, B, and C are matrices whose sizes are such that $(AB)C$ makes sense. Prove that $A(BC)$ makes sense and that $(AB)C = A(BC)$.

2. (Axler 3.C.4) Suppose $v_1, \ldots, v_m$ is a basis of V and W is finite-dimensional. Suppose $T \in \mathcal{L}(V, W)$. Prove that there exists a basis $w_1, \ldots, w_n$ of W such that all the entries in the first column of $\mathcal{M}(T)$ (with respect to the bases $v_1, \ldots, v_m$ and $w_1, \ldots, w_n$) are 0 except for possibly a 1 in the first row, first column.

3. (Axler 3.D.14) Suppose $v_1, \ldots, v_n$ is a basis of V. Prove that the map $T : V \to \mathbb{F}^{n,1}$ defined by $Tv = \mathcal{M}(v)$ is an isomorphism of V onto $\mathbb{F}^{n,1}$; here $\mathcal{M}(v)$ is the matrix of $v \in V$ with respect to the basis $v_1, \ldots, v_n$.

4. (Axler 3.D.18) Show that V and $\mathcal{L}(\mathbb{F}, V)$ are isomorphic vector spaces.

5. (Axler 3.D.9) Suppose V is finite-dimensional and $S, T \in \mathcal{L}(V)$. Prove that ST is invertible if and only if S and T are invertible.

6. (Axler 3.F.6a) Suppose V is finite-dimensional and $v_1, \ldots, v_m \in V$. Define a linear map $\Gamma : V' \to \mathbb{F}^m$ by
$$\Gamma(\phi) = (\phi(v_1), \ldots, \phi(v_m)).$$
Prove that $v_1, \ldots, v_m$ spans V if and only if Γ is injective.