MATH 110: Linear Algebra

Homework 6

Instructor Franny Dean

Instructions: Please type your solutions to the following in LaTeX and upload your solutions to Gradescope by 4:10pm on **Wednesday, July 26, 2023**. You are highly encouraged to work with your classmates, but your write up must be done independently without looking at any other student's solutions. One of these 6 questions will not be graded.

1. (Axler 5.B.2) Suppose $T \in \mathcal{L}(V)$ and $(T - 2I)(T - 3I)(T - 4I) = 0$. Suppose λ is an eigenvalue of T. Prove that $\lambda = 2$ or $\lambda = 3$ or $\lambda = 4$.

2. (Axler 5.C.16) The Fibonnacci sequence $F_1, F_2, \ldots$ is defined by

$$F_1 = 1, F_2 = 1, \text{ and } F_n = F_{n-2} + F_{n-1} \text{ for } n \geq 3.$$

   Define $T \in \mathcal{L}(\mathbb{R}^2)$ by $T(x, y) = (y, x + y)$.

   (a) Show that $T^n(0, 1) = (F_n, F_{n+1})$ for each positive integer n.

   (b) Find the eigenvalues of T.

   (c) Find a basis of $\mathbb{R}^2$ consisting of eigenvectors of T.

3. (Axler 6.A.5) Suppose $T \in \mathcal{L}(V)$ for finite-dimensional V is such that $||Tv|| \leq ||v||$ for every $v \in V$. Prove that $T - \sqrt{2}I$ is invertible.

4. (Axler 6.B.7) Find a polynomial $q \in \mathcal{P}_2(\mathbb{R})$ such that $p\left(\frac{1}{2}\right) = \int_0^1 p(x)q(x)dx$ for every $p \in \mathcal{P}_2(\mathbb{R})$.

5. (Axler 6.C.2) Suppose U is a finite-dimensional subspace of V. Prove that $U^\perp = \{0\}$ if and only if $U = V$.

6. (Axler 6.C.11) In $\mathbb{R}^4$, let $U = \text{span}((1, 1, 0, 0), (1, 1, 1, 2))$. Find $u \in U$ such that $||u - (1, 2, 3, 4)||$ is as small as possible.