# MATH 110: Linear Algebra

## Homework 7

Instructor Franny Dean

*Instructions:* Please type your solutions to the following in LaTeX and upload your solutions to Gradescope by 4:10pm on **Wednesday, August 2, 2023**. You are highly encouraged to work with your classmates, but your write up must be done independently without looking at any other student's solutions.

1. (Axler 7.A.2,4)
   (a) S'pose $T \in \mathcal{L}(V)$ and $\lambda \in \mathbb{F}$. Prove that $\lambda$ is an eigenvalue of $T$ if and only if $\bar{\lambda}$ is an eigenvalue of $T^*$.
   (b) S'pose more generally, $T \in \mathcal{L}(V, W)$. Prove that $T$ is injective if and only if $T^*$ is surjective. This implies that $T$ is surjective if and only if $T^*$ is injective. Why?
2. (Axler 7.A.15) Fix $u, x \in V$. Define $T \in \mathcal{L}(V)$ by $Tv = \langle v, u \rangle x$ for every $v \in V$.
   (a) Suppose $\mathbb{F} = \mathbb{R}$. Prove that $T$ is self-adjoint if and only if $u, x$ are linearly dependent.
   (b) Prove that $T$ is normal if and only if $u, x$ is linearly dependent.
3. (Axler 7.B.9,11)
   (a) Suppose $V$ is a complex inner product space. Prove that every normal operator has a square root.
   (b) Prove or give a counterexample: every self-adjoint operator on $V$ over $\mathbb{C}$ or $\mathbb{R}$ has a cube root. (An operator $S \in \mathcal{L}(V)$ is called a cube root of $T \in \mathcal{L}(V)$ if $S^3 = T$.)
4. (Axler 7.B.6) Prove that a normal operator on a complex inner product space is self-adjoint if and only if its eigenvalues are real.
5. (Axler 7.C.2) Suppose $T$ is a positive operator on $V$. Suppose $v, w \in V$ such that $Tv = w$ and $Tw = v$. Prove $v = w$.