MATH 110: Linear Algebra

Homework 8

Instructor Franny Dean

Instructions: Please type your solutions to the following in LaTeX and upload your solutions to Gradescope by 4:10pm on **Wednesday, August 9, 2023**. You are highly encouraged to work with your classmates, but your write up must be done independently without looking at any other student's solutions. Five of the six problems will be graded. Solutions to all 6 will be emailed after class Wednesday.

1. (Axler 8.A.2) Define $T \in \mathcal{L}(\mathbb{C}^2)$ by $T(w, z) = (-z, w)$. Find the generalized eigenspaces corresponding to the distinct eigenvalues of T.

2. (Axler 8.A.10) Suppose that $T \in \mathcal{L}(V)$ is not nilpotent. Let $n = \dim V$. Show that $V = \text{null } T^{n-1} \oplus \text{range } T^{n-1}$.

3. (Axler 8.B.5) Suppose V is a complex vector space and $T \in \mathcal{L}(V)$. Prove that V has a basis consisting of eigenvectors of T if and only if every generalized eigenvector of T is an eigenvector of T.

4. (Axler 8.C.7) Suppose V is a complex vector space. Suppose $T \in \mathcal{L}(V)$ is such that $P^2 = P$. Prove that the characteristic polynomial of P is $z^m(z-1)^m$, where $m = \dim \text{null } P$ and $n = \dim \text{range } P$. *Hint: First prove $V =$ null $P \oplus$ range P.*

5. Find the Jordan normal form of each of the following operators on $\mathbb{C}^4$:
   (a) An operator whose characteristic polynomial is $(z-1)^4$ and whose minimal polynomial is $(z-1)^2$.
   (b) The operator whose matrix with respect to the standard basis is
   $$\begin{pmatrix} 3 & 4 & 5 & 0 \\ 0 & 2 & 6 & 7 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

6. (Axler 10.A.10) Suppose V is an inner product space and $T \in \mathcal{L}(V)$. Prove that $\text{trace } T^* = \overline{\text{trace } T}$.

Ungraded challenge: (Axler 8.B.11) Prove the following proposition from lecture: Suppose $T \in \mathcal{L}(V)$ and $\lambda \in \mathbb{F}$. Then for every basis of V with respect to which T has an upper triangular matrix, the number of times that λ appears on the diagonal of the matrix of T equals the multiplicity of λ as an eigenvalue of T.